Recursion

Martin Steffen

Autumn 2017

Recursion

- Latin: *recurrere*

> Linguistics/Mathematics:
> Relating to or involving
> the repeated application
> of a rule, definition, or
> procedure to successive
> results.
>
> Oxford Dictionary

Recursion

The reduction of a problem to a simpler problem

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Recursion

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> Oxford Dictionary

Recursion

The reduction of a problem to a (simpler) version of itself

- self-referentiality: recursive problems, algorithms, definitions, data structures, ...

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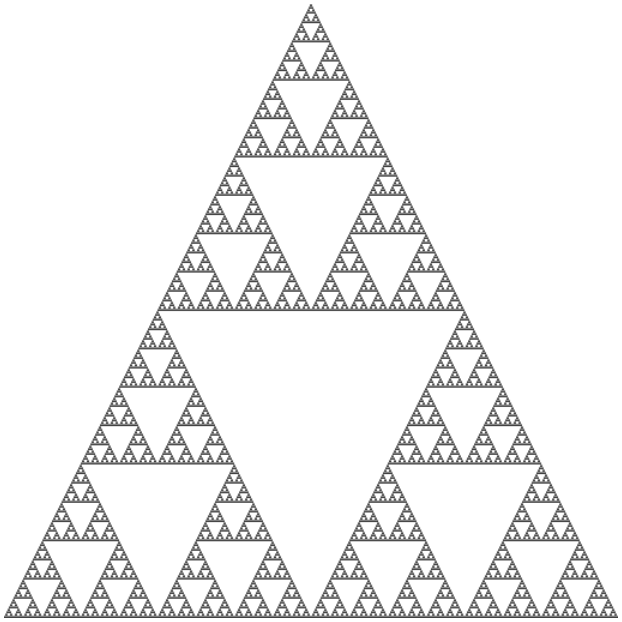
Recursion



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Multiplication

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From elementary school: $a \times b$

"a times b" means: add b onto itself, and do that a-times.

$$a \times b = \underbrace{b + b + \ldots + b}_{a \geq 0}$$

Example (4×3)

3
plus 3 gives 6,
plus 3 gives 9,
plus 3 gives 12, and done.

Multiplication

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"a times b" means: add b onto itself, and do that a-times.

$$a \times b = 0 + \underbrace{b + b + \ldots + b}_{a \geq 0} = \sum_{i=1}^{a} b$$

Example (4×3)

3

plus 3 gives 6,

plus 3 gives 9,

plus 3 gives 12, and done.

As static method

```
public class Timesiter {
    public static long times_iter(int a, int b) {
        int r = 0;
        for  (int i = 1; i<= a;i++) {
            r = r + b;
        };
        return r;
    }

  public static void main(String args[]) {
      System.out.println(times_iter (4, 3));
  }
}
```

As static method

```
public class Timesiter {
    public static long times_iter(int a, int b) {
        int r = 0;
        for  (int i = 1; i<= a;i++) {
            r = r + b;
        };
        return r;
    }

  public static void main(String args[]) {
      System.out.println(times_iter (4, 3));
  }
}
```

As static method (2)

```
public static long times_iter(int a, int b) {
    int r = 0;
    for  (int i = 1; i<= a;i++) {
        r = plus(r,b);
    };
    return r;
}
public static int plus(int x, int y) {
    return x + y;}
```

Just to focus

```
times_iter(a, b) {
  int r = 0;
  for  (i = 1; i<= a;i++) {
       r = plus(r,b);  };
  return r;
}

plus(x, y) { return x + y;}
```

$$a \times b = 0 + \underbrace{b + b + \ldots + b}_{a \geq 0} = \sum_{i=1}^{a} b$$

Multiplication: one more time

$$a \times b = 0 + \underbrace{b + b + \ldots + b}_{a \geq 0} = \sum_{i=1}^{a} b$$

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Multiplication: one more time

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$$a \times b = b + \underbrace{b + \ldots + b}_{a-1 \text{ “mal” b}} \quad = \quad b + ((a-1) \times b)$$

Self-referential definition (= recursive)

Multiplication calculated with the help of Multiplication (and addition)

Multiplication: one more time

$$a \times b = b + \underbrace{b + \ldots + b}_{a-1 \text{ “mal” b}} \quad = \quad b + ((a-1) \times b)$$

Self-referential definition (= recursive)

Multiplication of $a \times b$ calculated with the help of multiplication of $a - 1 \times b$ (and addition)

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In Java

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```
public static long times(int a, int b) {
    if (a == 0) {
        return 0;
    } else {
        return b + times (a-1, b);
    }
}
```

In Java

```
times(a, b) {
        if  (a == 0) {
            return 0;
        } else {
            return b + times (a-1, b);
        }
    }
```

$$a \times b = b + ((a-1) \times b)$$

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In Java

```
times(a, b) {
        if  (a == 0) {
            return 0;
        } else {
            return b + times (a-1, b);
        }
    }
```

$$a \times b \Leftarrow b + ((a-1) \times b)$$

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And what about negative numbers?

```
public static long times(int a, int b) {
    if (a >= 0) {
        if (a == 0) {
            return 0;
        } else {
            return b + times (a-1, b);
        }
    } else {
        return - (times (-a, b));
    }
}
```

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Take home message

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Recursion

A recursive method is defined by via itself (= calling itself).

Take home message

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Recursion

A recursive method is defined by via itself (= calling itself).

"Proper recursion" (= termination)

To solve a problem:

1. define a method via calling itself of a "simpler" version of the problem.
2. and: there is a simplest problem which is directly solvable (= without further recursion), such that the recursion "finds the exit" and terminates

Take home message

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Recursion

A recursive method is defined by via itself (= calling itself).

"Proper recursion" (= termination)

To solve a problem:

1. define a method via ~~an~~ applications of itself on ~~one~~ "simpler" versions of the problem
2. and: there ~~is one~~ are simplest problems, which ~~is~~ are directly (= without further recursion) solvable, such that the recursion "finds the exit" and terminates.

Factorial

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Iteratively: "1 times 2 times 3... until n"

$$n! = 1 \times 2 \times \ldots \times (n-1) \times n = \prod_{i=1}^{n} i$$

Factorial

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Iteratively: "1 times 2 times 3... until n"

$$n! = 1 \times 2 \times \ldots \times (n-1) \times n = \prod_{i=1}^{n} i$$

```
factorial_iter(n) {
  long result = 1;
  for  (i = 1; i<= n;i++) {
     result = result * i;
  };
  return result;
```

One more time, but recursive

$$n! = n \times (n-1) \times \ldots \times 2 \times 1$$

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One more time, but recursive

$$n! = n \times \underbrace{(n-1) \times \ldots \times 2 \times 1}_{(n-1)!}$$

```
factorial(n) {
    if (n == 1) return 1;
    return n * factorial(n-1);
}
```

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Side by side comparison

```
factorial_iter(n) {
  long result = 1;
  for (i = 1; i<= n;i++) {
     result = result * i;
  };
  return result;
}
```

- variable `result` updated step by step (= iteratively)

```
factorial(n) {
   if (n == 1) return 1;
   return n * factorial(n-1);
}
```

- variable `result` local local to method body
- exactly *one* value per call

Side by side comparison

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```

- variable `result` local
  local to method body
- exactly *one* value per call

At run time

$fac(5)$

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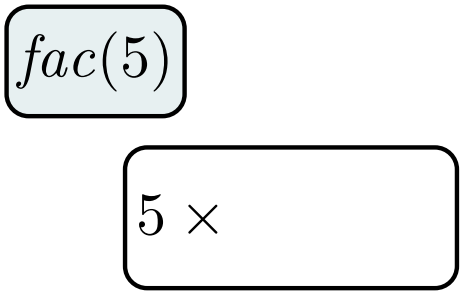
At run time



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At run time

$$\boxed{fac(5)}$$

$$\boxed{5 \times \boxed{fac(4)}}$$

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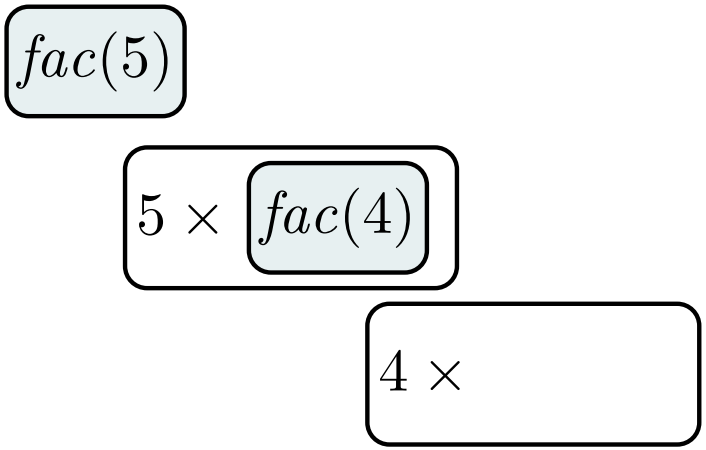
At run time



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At run time

$$fac(5)$$

$$5 \times fac(4)$$

$$4 \times fac(3)$$

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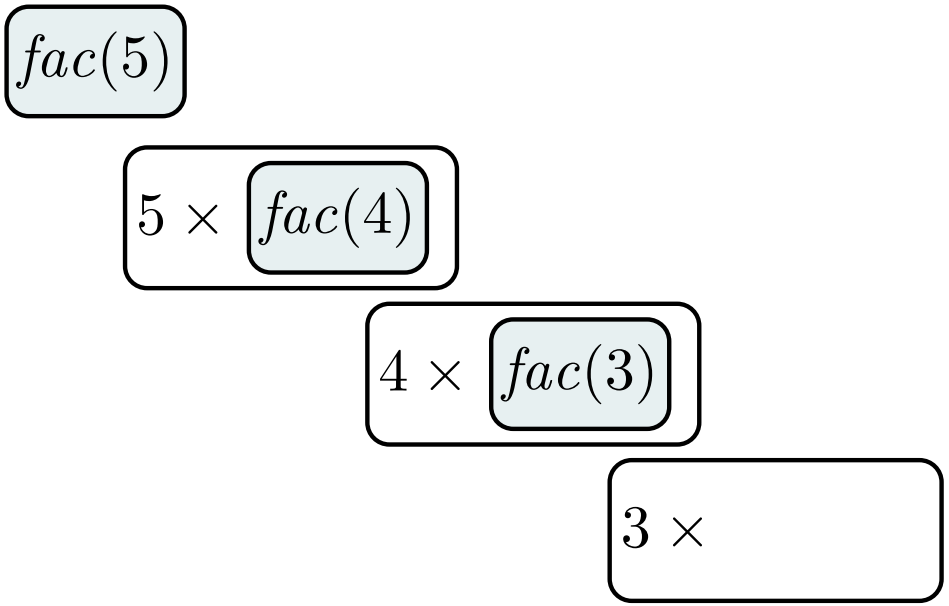
At run time



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At run time

$fac(5)$

$5 \times fac(4)$

$4 \times fac(3)$

$3 \times fac(2)$

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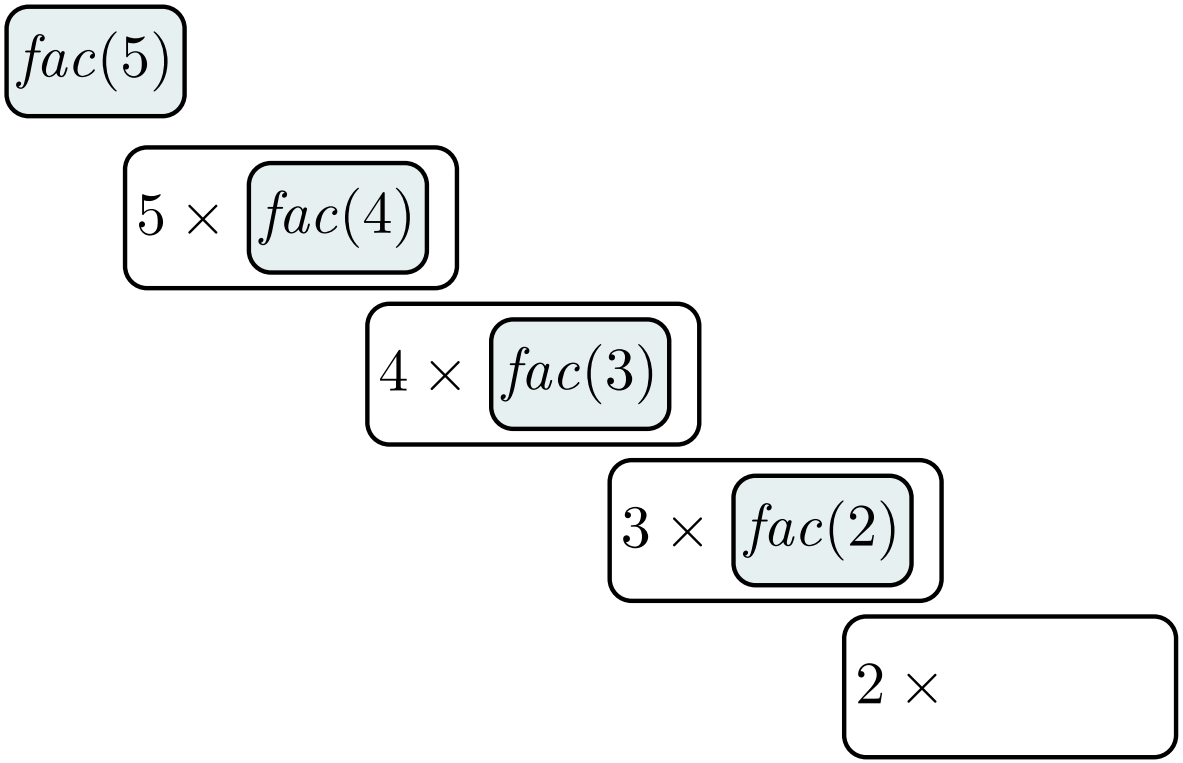
At run time



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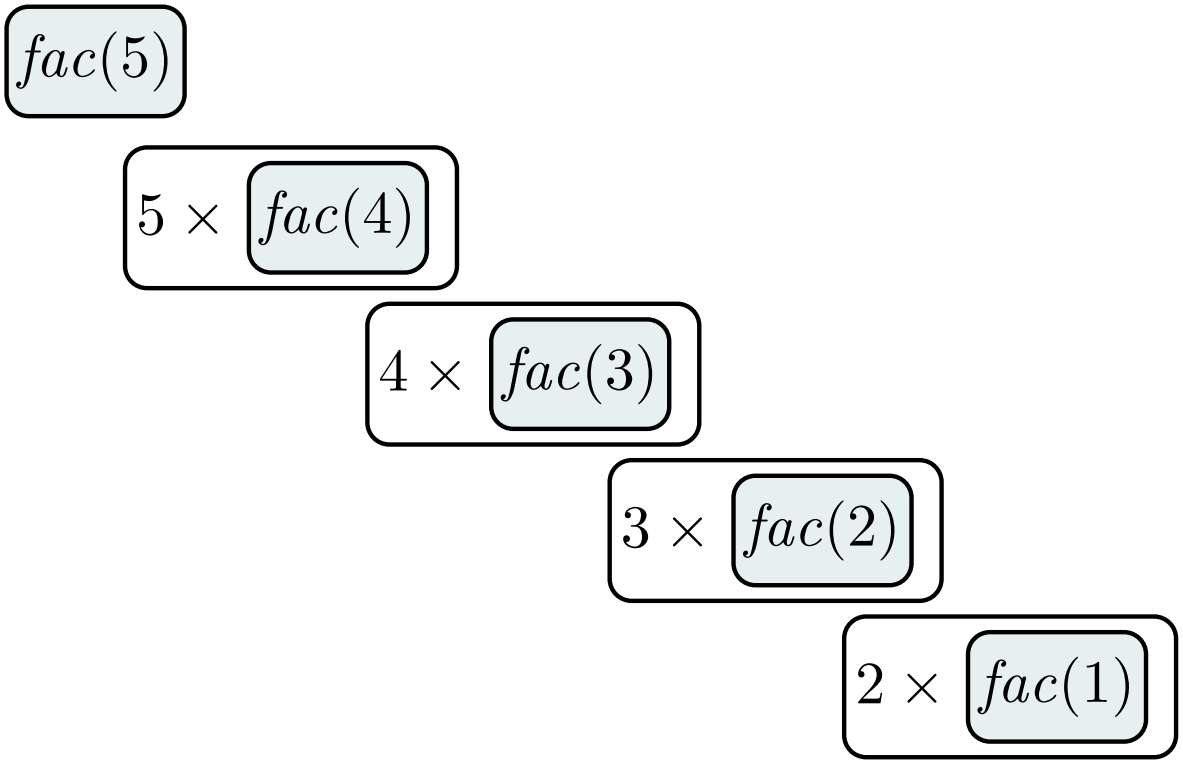
At run time



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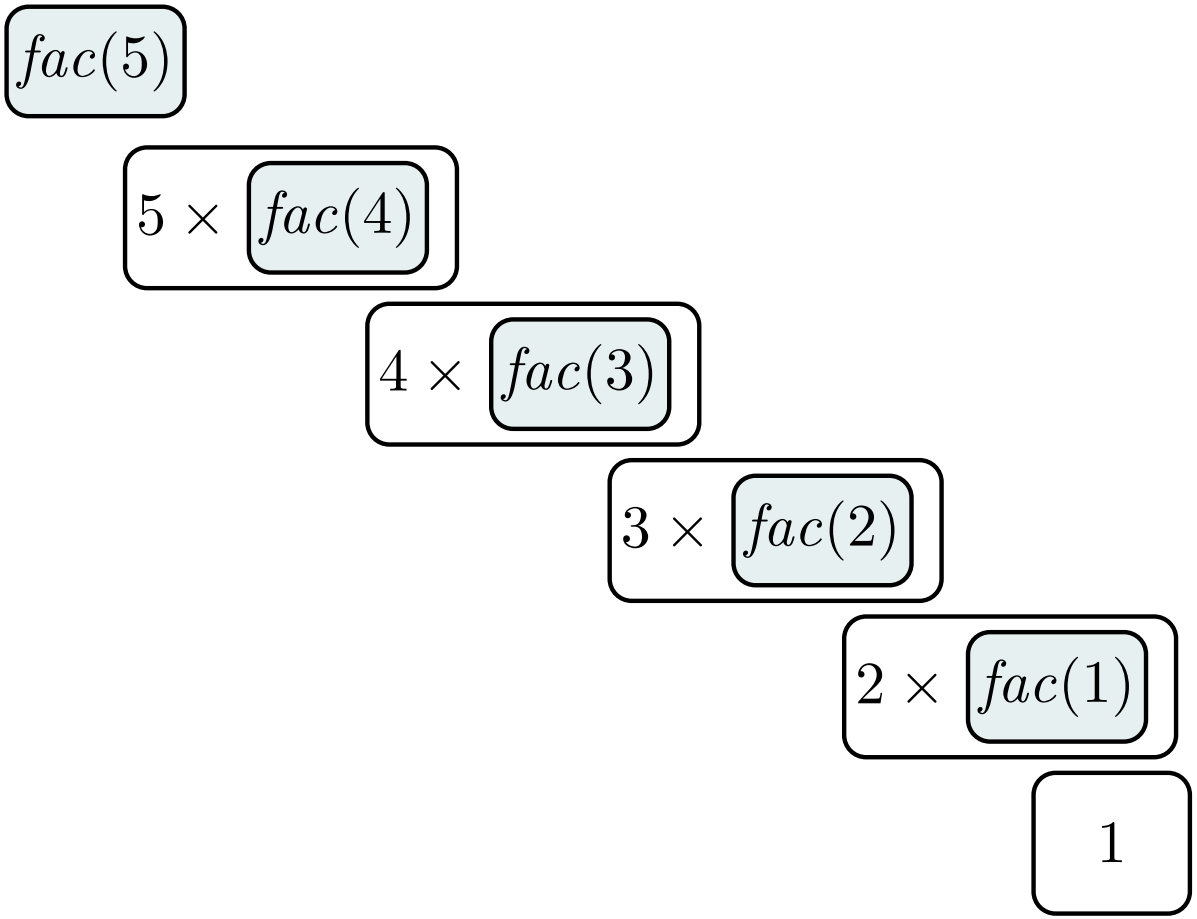
At run time



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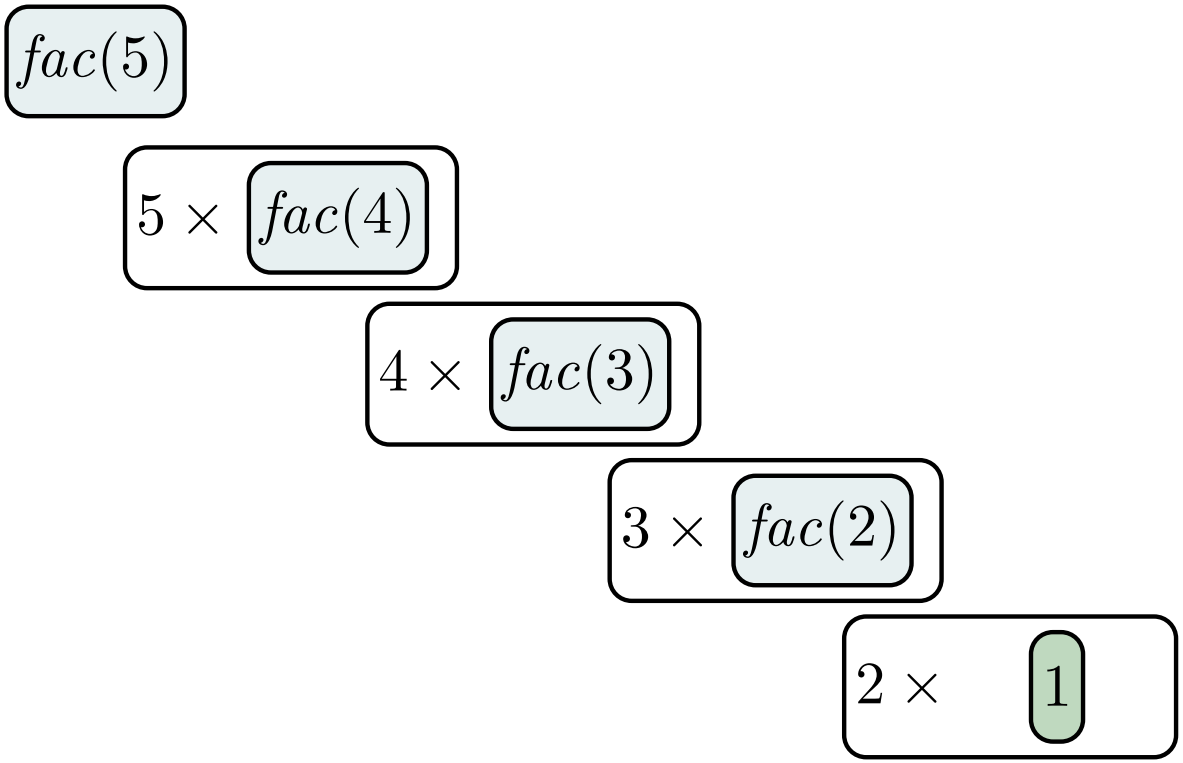
At run time



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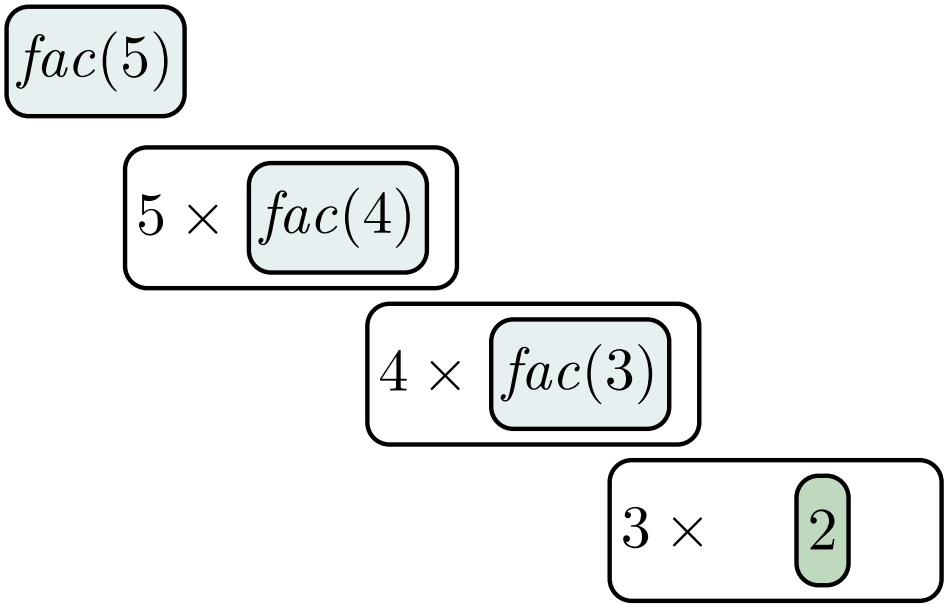
At run time



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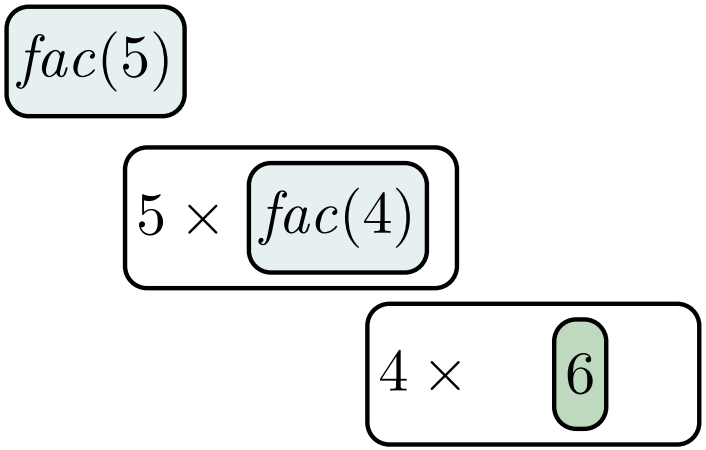
At run time



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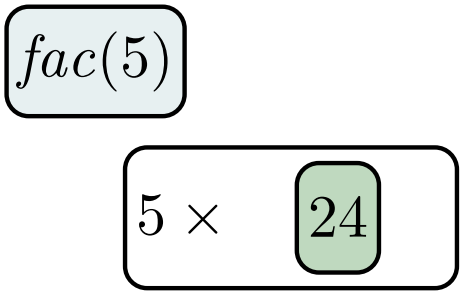
At run time



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At run time

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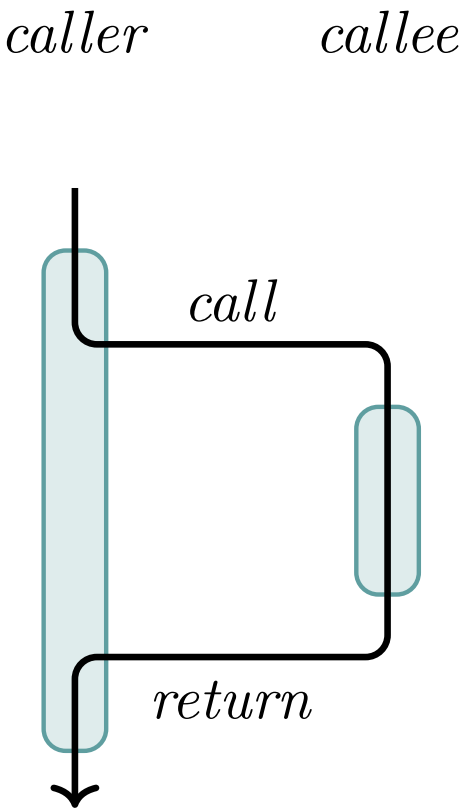
Caller and callee



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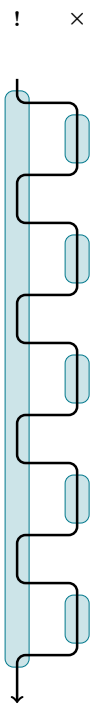
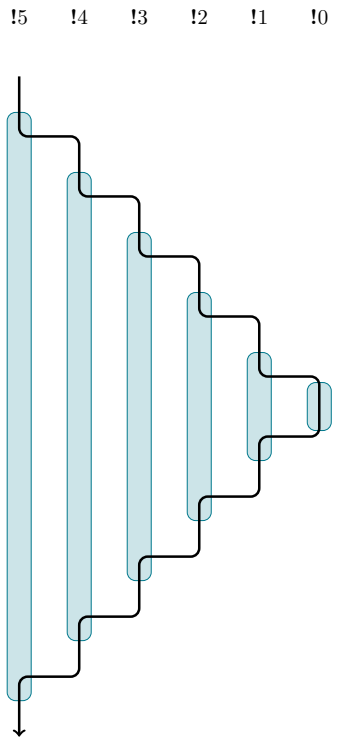
Life time of local variables





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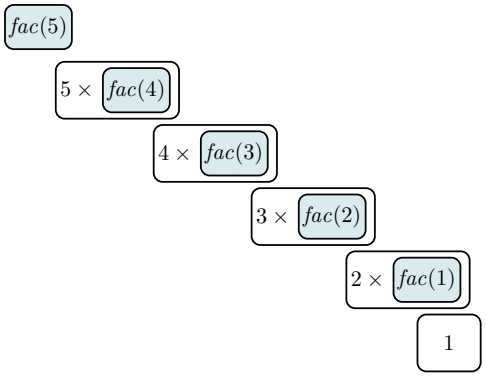
Run time stack




```
factorial(n) {
    long result;
    if (n == 1) { result = 1; }
    else {
      result = n * factorial(n-1);
    };
    return result;
}
```

- $n!$: n (here 5) incarnations of `result`
- allocation/deallocation: LIFO $\Rightarrow$ run-time stack
- *dynamic* memory management

Leonardo da Pisa

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The rabbit problem

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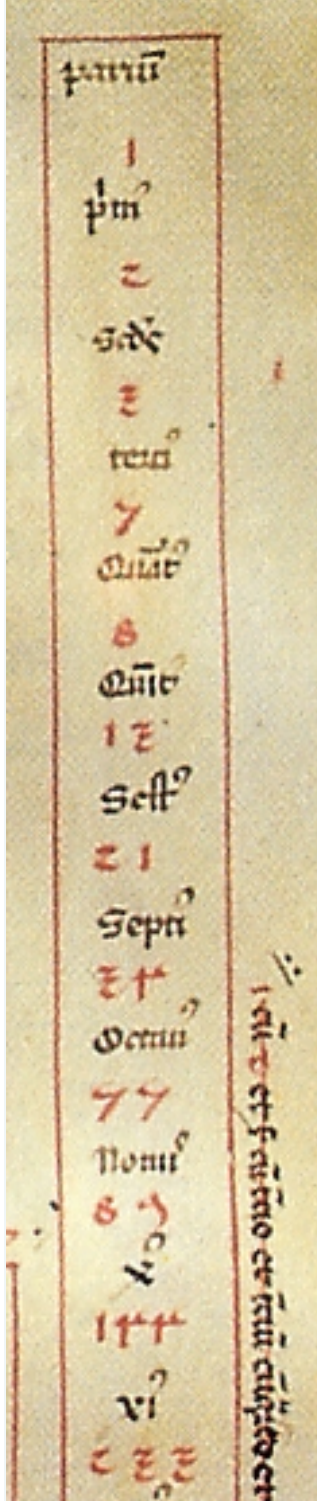
1. A rabbit grows up in *one* month
2. Each grown up pair of rabitt breeds a pair of rabitts each month

Question

Starting with one pair, how many pairs do we have after n months

Fibonacci's solution

Month	rabbit pairs		
	newborn	grown up	total
0	1	0	1
1	0	1	1
2	1	1	2
3	1	2	3
4	2	3	5
5	3	5	8
	⋮		



Fibonacci's solution

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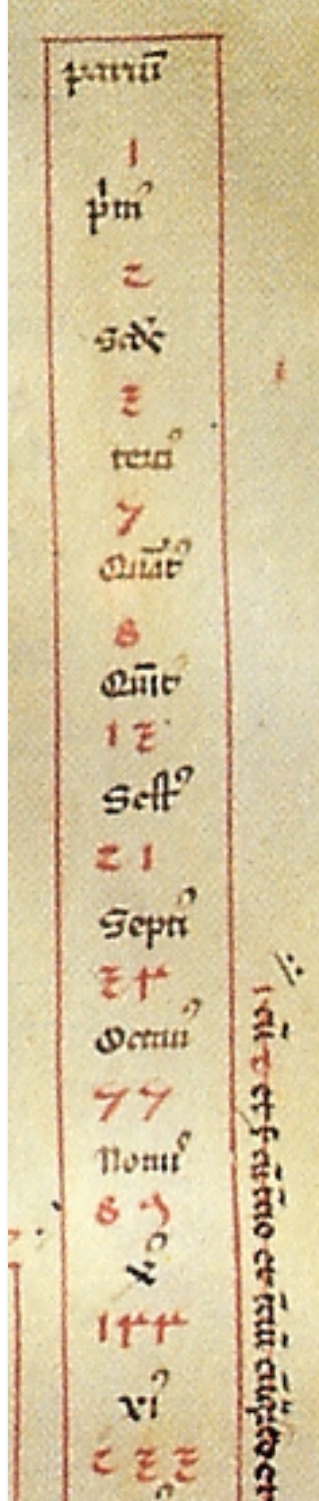
3	1	2	3
4	2	3	5
5	3	5	8
	⋮		

$$f_n = \begin{cases} 1 & \text{if } n = 0 \text{ or } n = 1 \\ f_{n-1} + f_{n-2} & \text{otherwise} \end{cases}$$



In Java

```
public static int fibonacci(int n) {
  if (n == 0) return 1;                    // base case
  if (n == 1) return 1;                    // base case
  return
    fibonacci (n-1) + fibonacci(n-2); // induction case
  }
```

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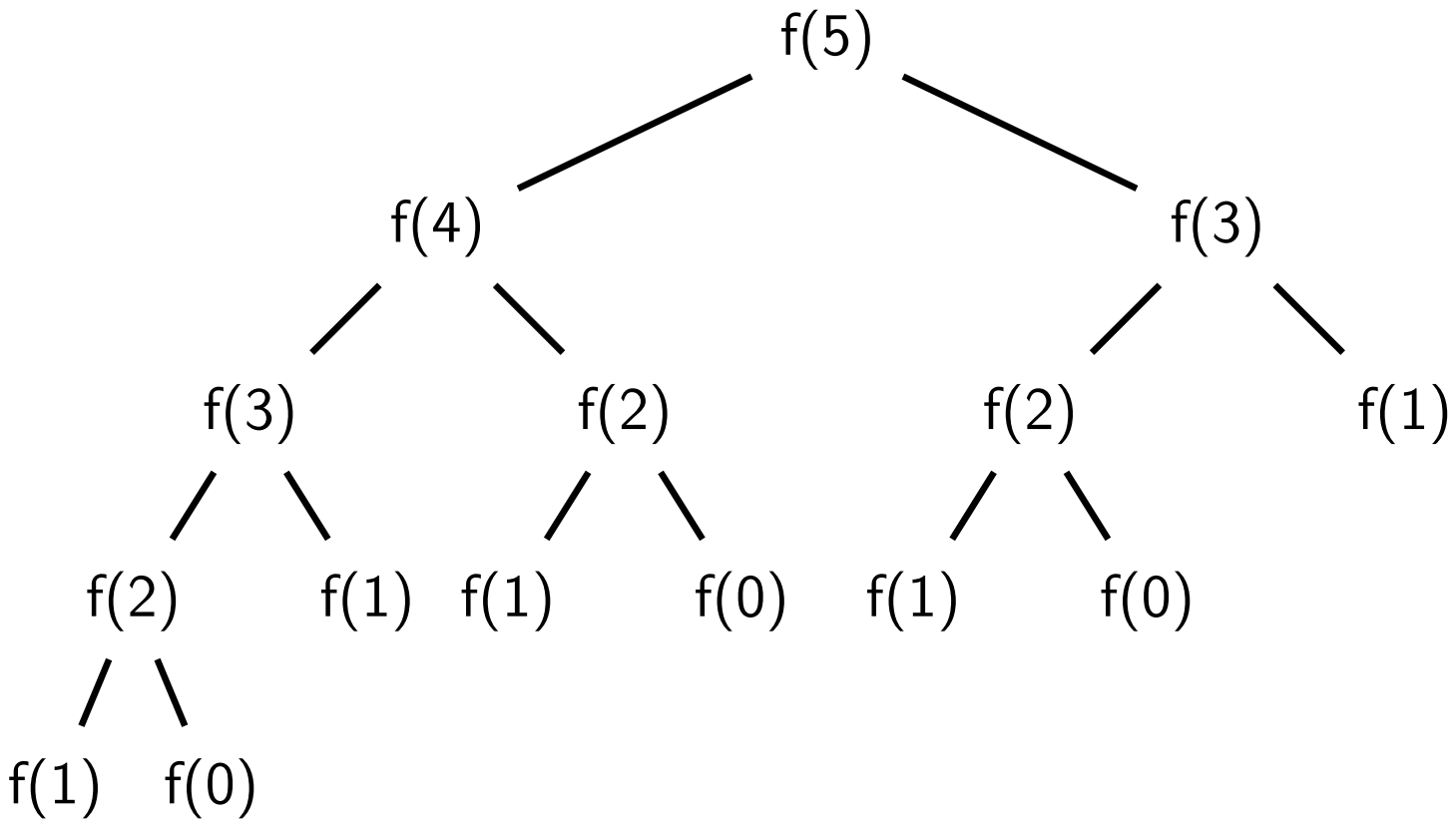
Calls



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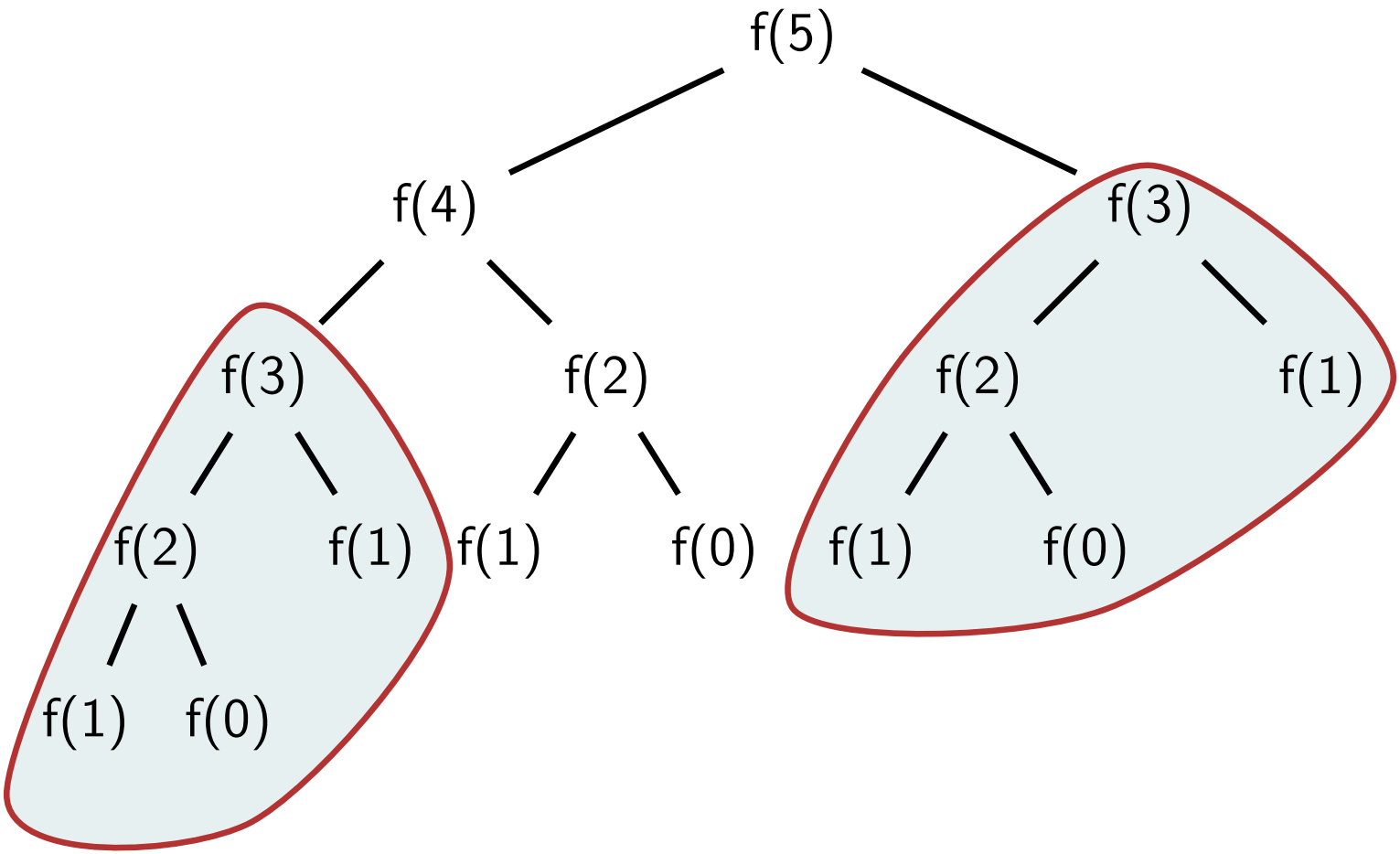
Calls



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As an aside . . .

```
\node{f(5)}
child{node {f(4)}
  child{node {f(3)}
    child{node {f(2)}
      child{node {f(1)}}
      child{node {f(0)}}
    }
    child{node {f(1)}}
  }
  child{node {f(2)}
    child{node {f(1)}}
    child{node {f(0)}}
  }
}
child{node {f(3)}
  child{node {f(2)}
    child{node {f(1)}}
    child{node {f(0)}}
  }
  child{node {f(1)}}
}
;
U:---  fibonaccitree.tex
```

tree = rec. data structure

tree node =

- node without children (“leaf”), or
- with n (here 2) tree nodes as children

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Trees


```
public class TreeO {
    private TreeO left, right = null;
    private Object data;
```

in practice: tree structure mostly more complex

- list of childred/sub-trees
- instead of `Object`: *"generics"*
- further methods, constructors, pointers
- avoid null-pointer
- interfaces
- ...

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We can do more efficiently

```
public static int fibonacci(int n, int a, int b) {
    if (n == 0) return b;     // let's start with 1
    return fibonacci (n-1,b,a+b);
}
```

- a = "newborn", b = "grown up"
- initial call with `fibonacci`(n, 0, 1)

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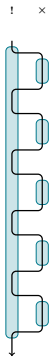
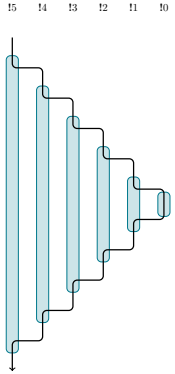
Life time of local variables

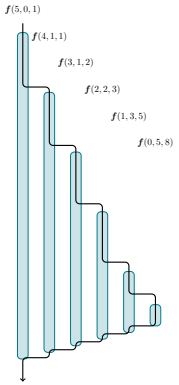
Fac. iteratively



Fac. recursively



Fib. recursively




```
public static int fibonacci(int n, int a, int b) {
    if (n == 0) return b;     // let's start with 1
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```

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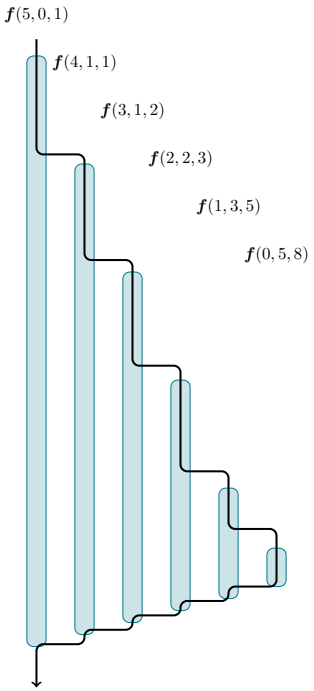
Tail recursion



- recursive call: last in the method body

```
public static int fibonacci(int n, int a, int b) {
    if (n == 0) return b;     // let's start with 1
    return fibonacci (n-1,b,a+b);
}
```

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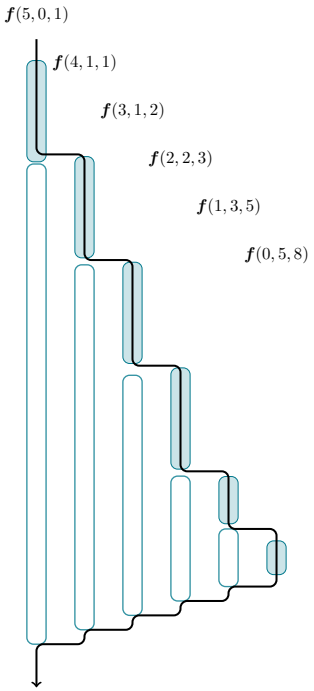
Tail recursion



- recursive call: last in the method body
- consequently:
  - Stack unnecessary ...
  - iterative-rekursively
- often: compiler optimization (not in Java, but on the language's todo-list)

```
public static int fibonacci(int n, int a, int b) {
    if (n == 0) return b;     // let's start with 1
    return fibonacci (n-1,b,a+b);
}
```

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“tail-recursive” calls?

```
public static int fibonacci(int n, int a, int b) {
    if (n != 0) return fibonacci (n-1,b,a+b);
    return b;
}
```

```
public static long factorial(int n) {
    if (n == 1) return 1;
    return n * factorial(n-1);
}
```

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Rekursion vs. Iteration

LEARN ERLANG

absolute beginners

Erlang Tutorial

- Erlang - Home
- Erlang - Overview
- Erlang - Environment
- Erlang - Basic Syntax
- Erlang - Shell
- Erlang - Data Types
- Erlang - Variables
- Erlang - Operators
- Erlang - Loops
- Erlang - Decision Making
- Erlang - Functions
- Erlang - Modules
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- Erlang - Numbers
- Erlang - Strings
- Erlang - Lists
- Erlang - File I/O
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- Erlang - Maps

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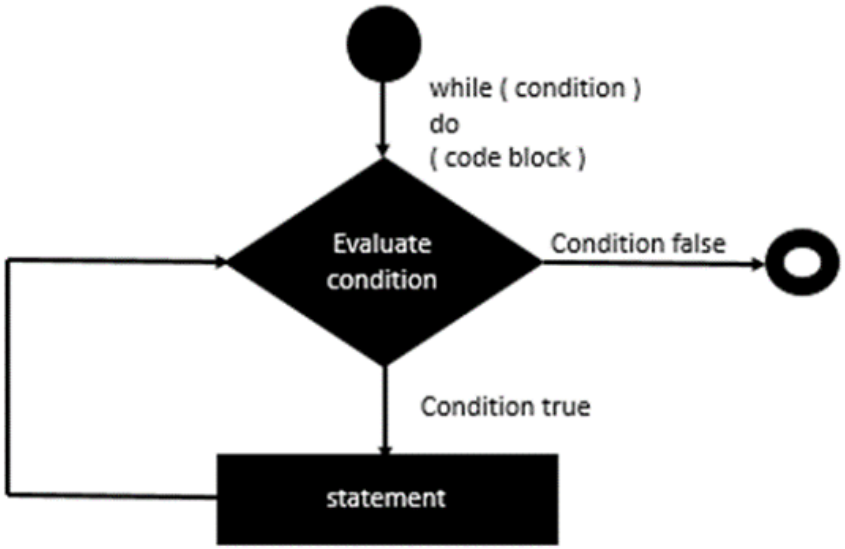
Erlang is a functional programming language and what needs to be remembered about all functional programming languages is that they don't offer any constructs for loops. Instead, functional programming depends on a concept called recursion.

while Statement Implementation

Since there is no direct while statement available in Erlang, one has to use the recursion techniques available in Erlang to carry out a while statement implementation.

We will try to follow the same implementation of the while loop as is followed in other programming languages. Following is the general flow which will be followed.



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Rekursion vs. Iteration

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- Recursion and iteration: in principle *equally expressive*

Recursion, only interesting for “number theoreticians”?

Git: Merge

```
[msteffen@rijkaard mmgo]$ git pull
remote: Counting objects: 9, done.
remote: Compressing objects: 100% (2/2), done.
remote: Total 9 (delta 7), reused 9 (delta 7), pack-reused 0
Unpacking objects: 100% (9/9), done.
From github.com:dfava/favasynthesis
   0d9fa6c..154dd44  master     -> origin/master
Merge made by the 'recursive' strategy.
 papers/mmgo/intro.tex | 52 ++++++++++++++++++++++++++++++----------------------
 1 file changed, 30 insertions(+), 22 deletions(-)
[msteffen@rijkaard mmgo]$ ▌
```

Recursion, only interesting for “number theoreticians”?

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Internet DNS

Recursive DNS is essentially the opposite of Dyn Standard DNS which is an authoritative DNS service that allows others to find *your* domain while Recursive DNS allows *you* to resolve other people’s domains.

The Longer Answer

Recursive DNS provides recursive DNS. Yes, that’s recursive (something which repeats or refers back to itself) and confusing. In order to make a distinction between the service we provide and the general concept of recursive DNS, here’s an explanation.

To better illustrate how recursive DNS works, let’s imagine you are sitting at a computer in your study at home. You’re connected to the Internet by a cable connection and you are surfing the web looking for widgets. You have no idea where to find widgets, so you open your web browser and type in http://www.google.com.

Recursion, only interesting for “number theoreticians”?

Martin Steffen

The multi-million $ heist via recursion[1]

Deconstructing theDAO Attack: A Brief Code Tour

18 JUNE 2016 on thedao, security, ethereum, solidity

TheDAO was attacked today, and the attacker seems to have made off with 3.5mm ether (at time of writing in excess of $45mm). The vulnerability was the Race To Empty or Recursive Call attack.

[1] in virtual money (ether/“blockchain”) and temporary.

Search

Martin Steffen

Goal

- Input: integer-array + number
- Output: if number in the array: index *where*

3	12	21	23	42	48	50	55	57	60	62	67	75	79	89	91

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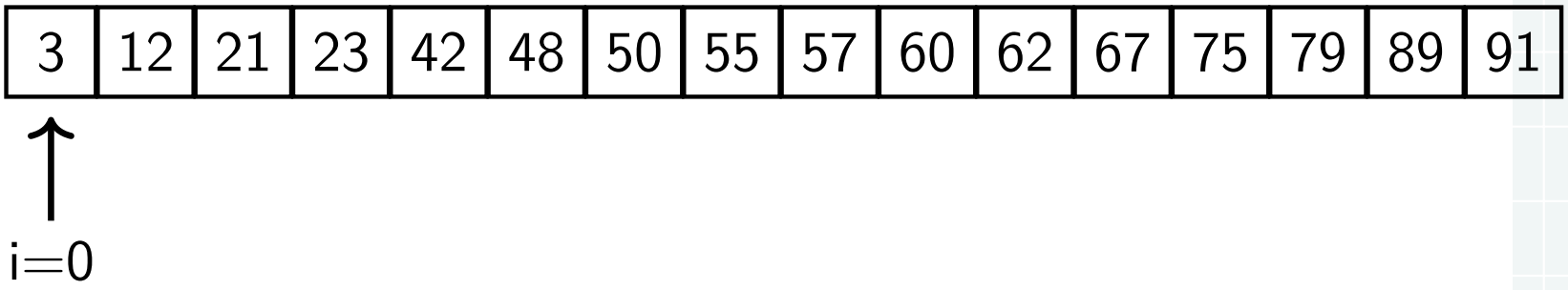

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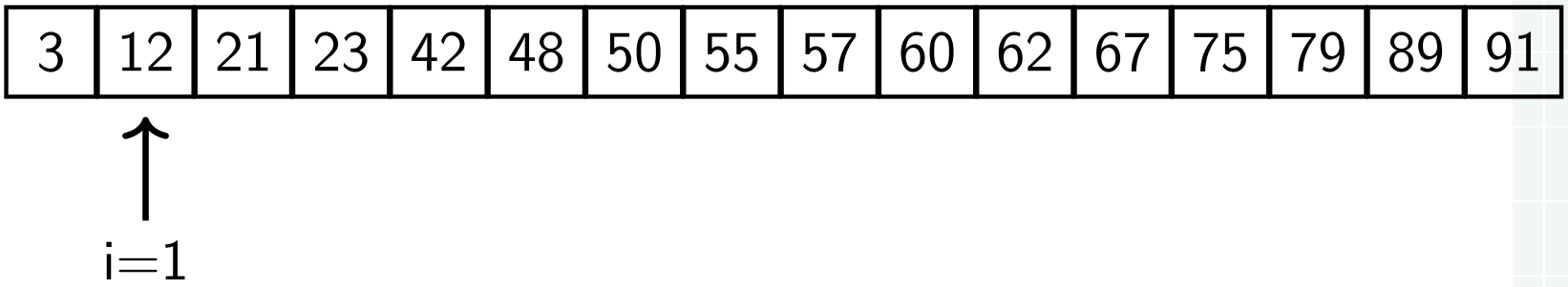

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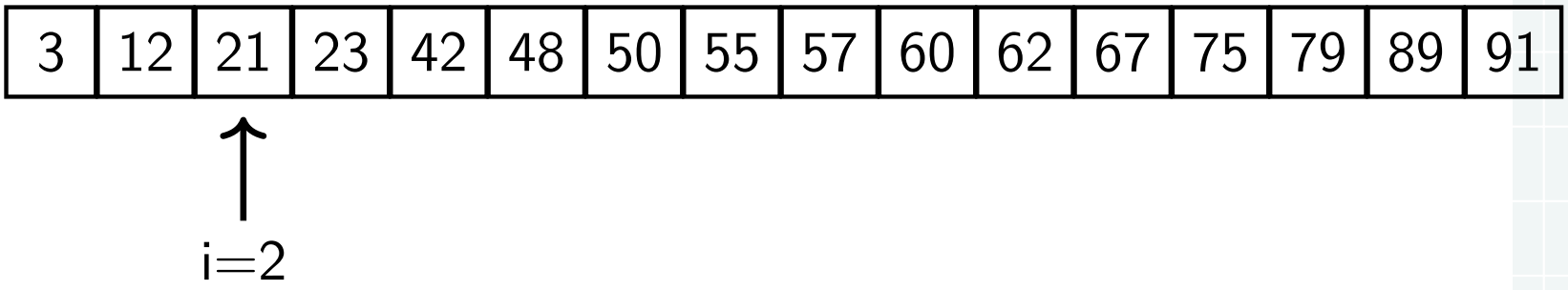

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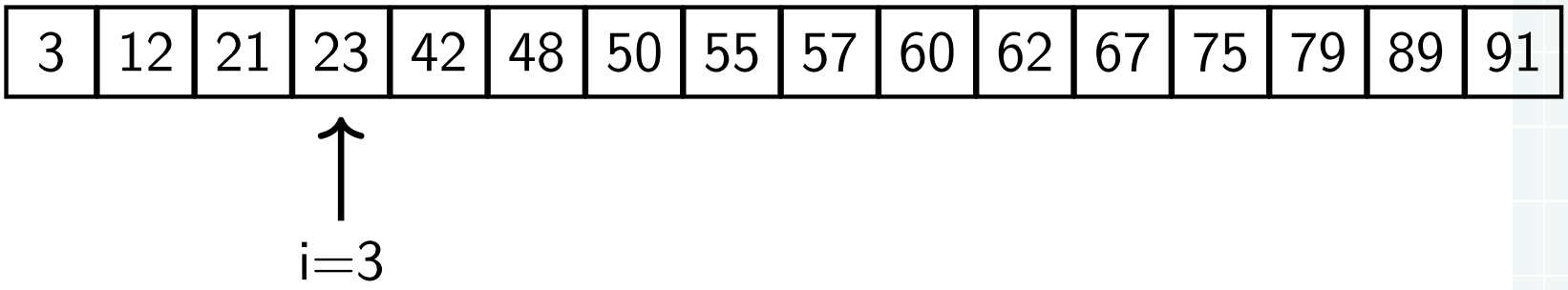

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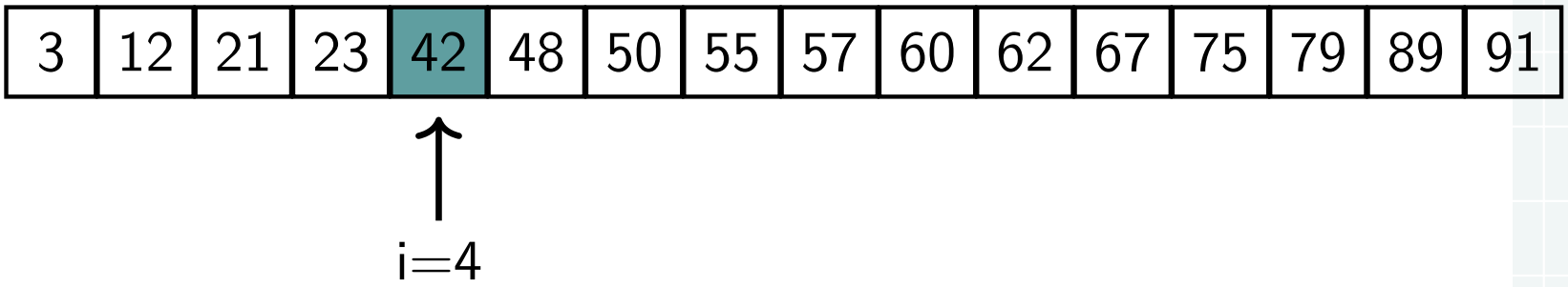

We can do better: binary search

Martin Steffen

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- Input: sorted Integer-array + number
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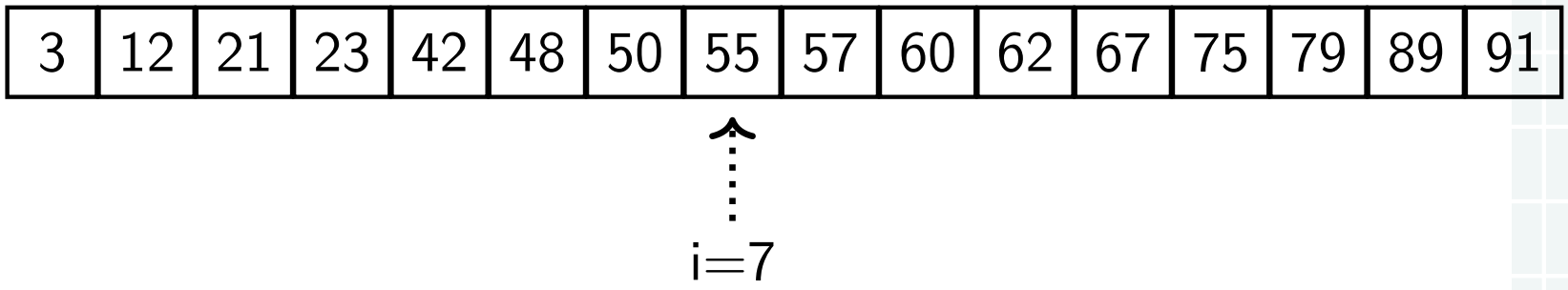

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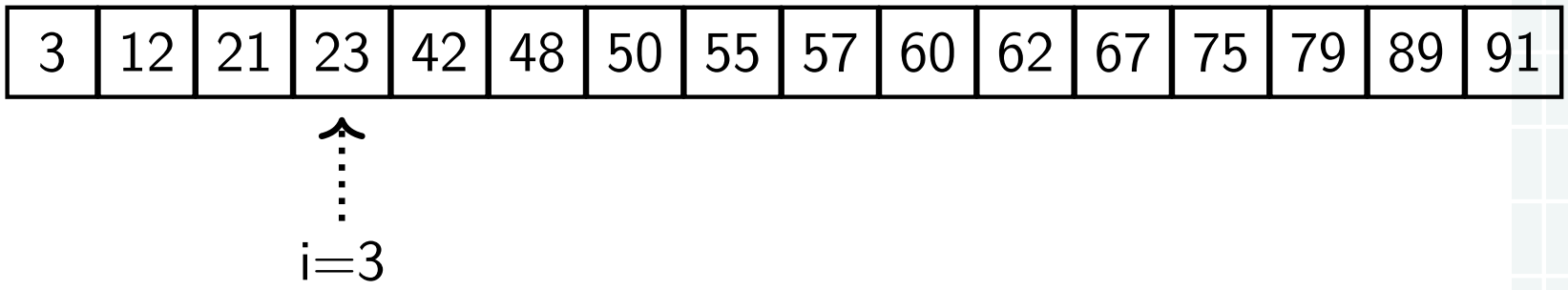

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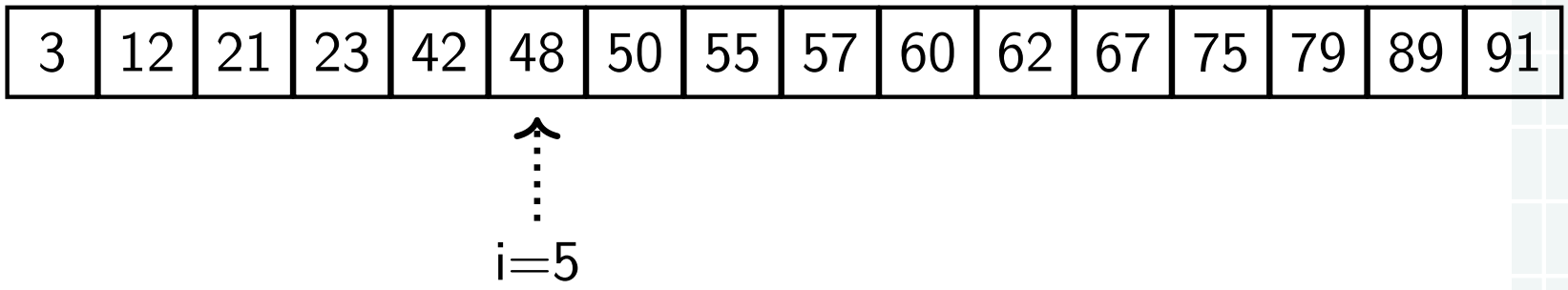

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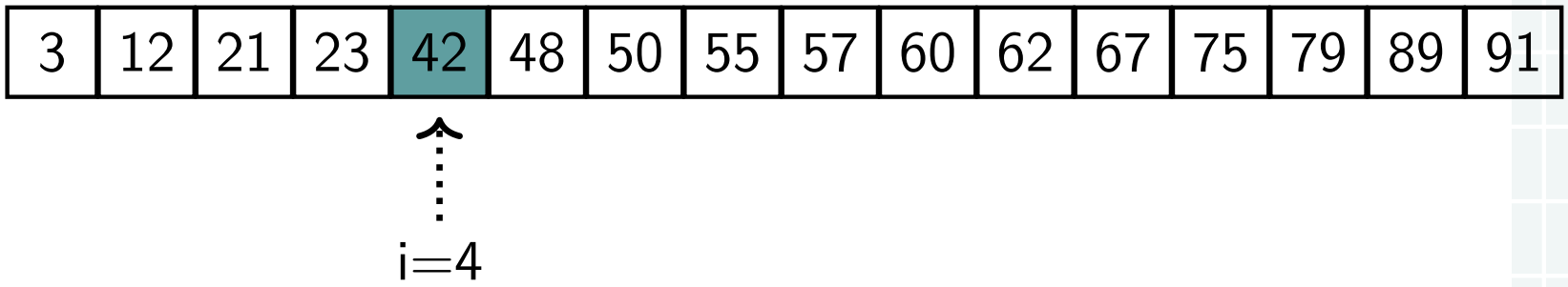

Recursive approach

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Divide & conquer: search for `elem`

- look up in the *middle* of the array
- if equal to `elem` $\Rightarrow$ done
- if smaller than `elem` $\Rightarrow$ search rekursively in the right half
- if larger than `elem` $\Rightarrow$ search rekursively in the left half

In Java

```
static int search (int elem, int[] a, int low, int high) {
    System.out.println(low);
    System.out.println(high);
    if (low == high) {
        if (elem == a[low]) {
            return low;
        } else {
            return -1;
        }
    } else {  // low ≠ high
        int m = (low + high) / 2;
        if (elem < a[m]) {
            return search(elem, a, low, m-1);
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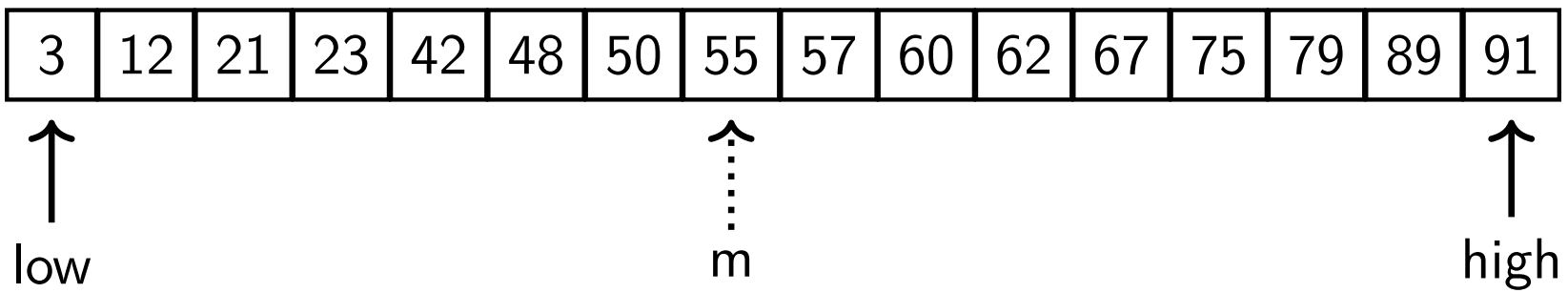

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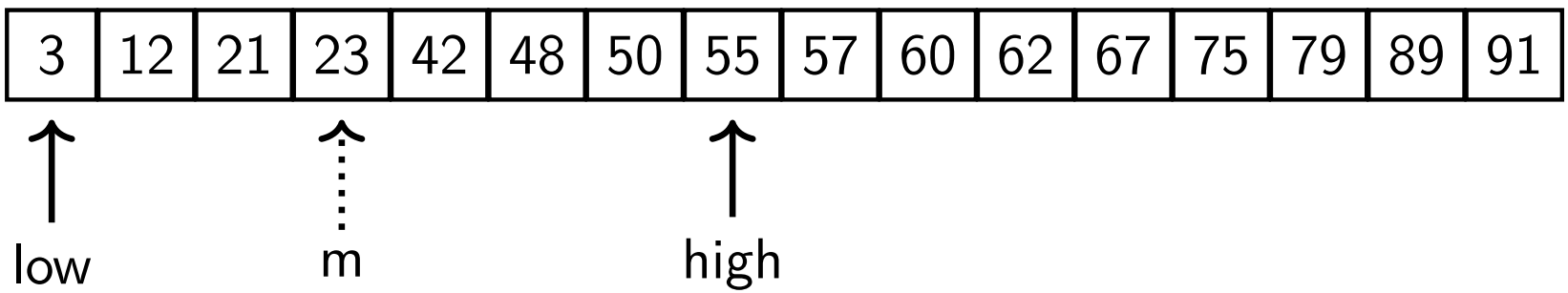

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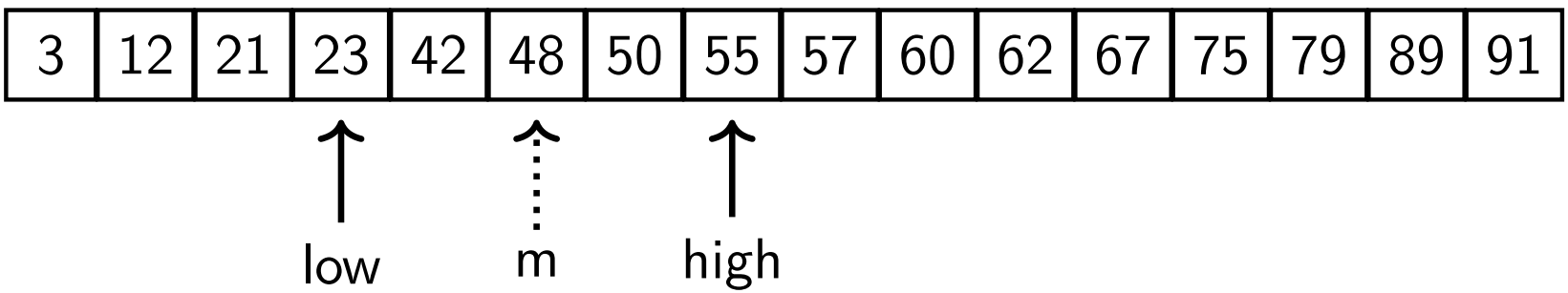

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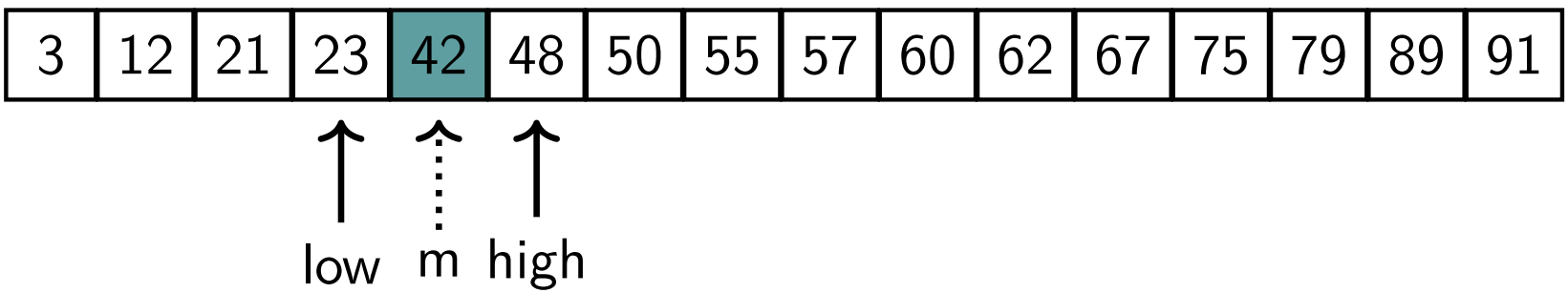

“Correctness”

Argument

1. The “divide-and-conquer” idea seems plausibly sound
2. Termination
   - each recursive call renders the problem smaller (induktion case)
   - there is a smallest problem (base case)

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1. The “divide-and-conquer” idea seems plausibly sound
2. Termination
   - each recursive call renders the problem smaller (induktion case)
   - there is a smallest problem (base case)
   - Alas: it only looks like that . . .

```
-*- mode: compilation; default-directory: "~/javaexamples/" -*-
Compilation started at Wed Oct 11 15:58:48

java Binsearch
Exception in thread "main" java.lang.StackOverflowError
        at Binsearch.search(Binsearch.java:12)
        at Binsearch.search(Binsearch.java:12)
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        at Binsearch.search(Binsearch.java:12)
-:%*-  *compilation*   Top L1     [(Compilation:exit [1] Abbrev)]
```

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On second thought . . .

```
static int search (int elem, int[] a, int low, int high) {
    if (low > high) return -1; // empty
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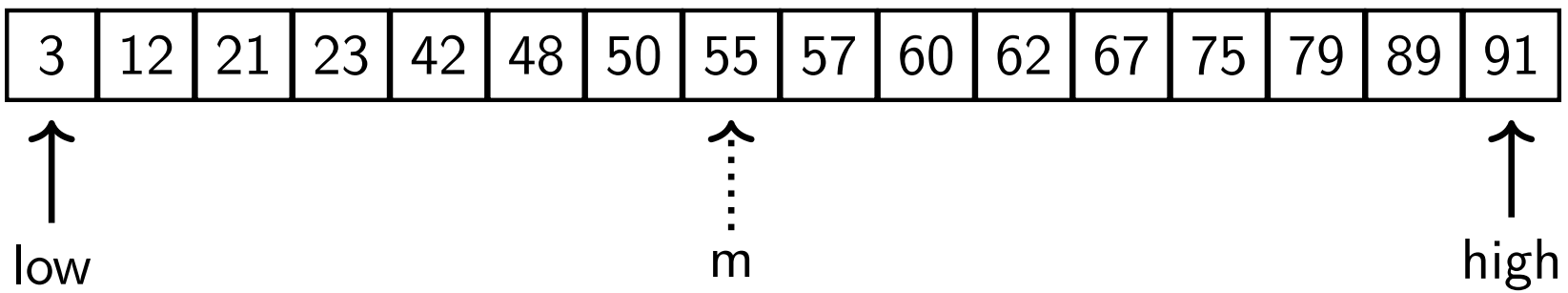

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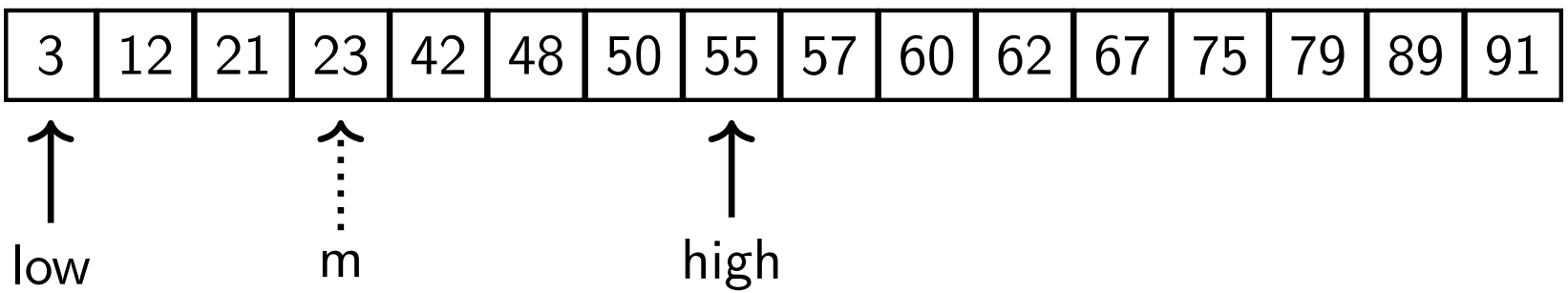

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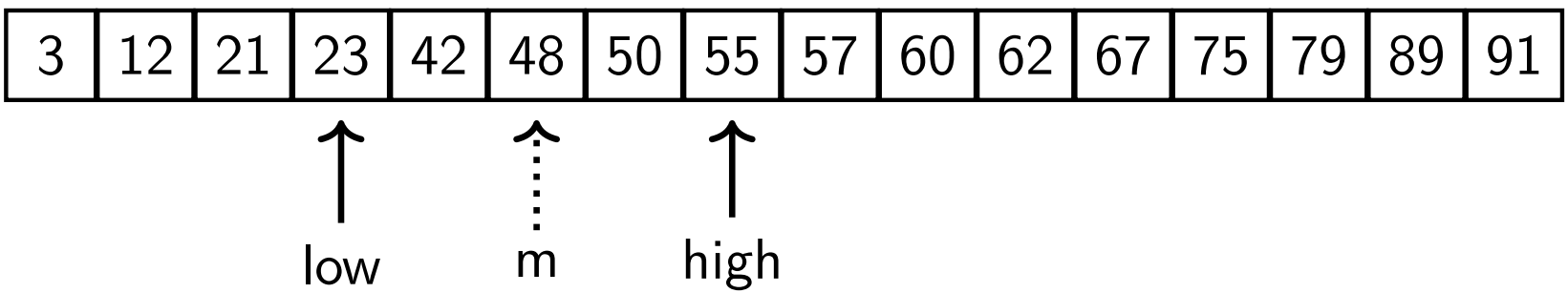

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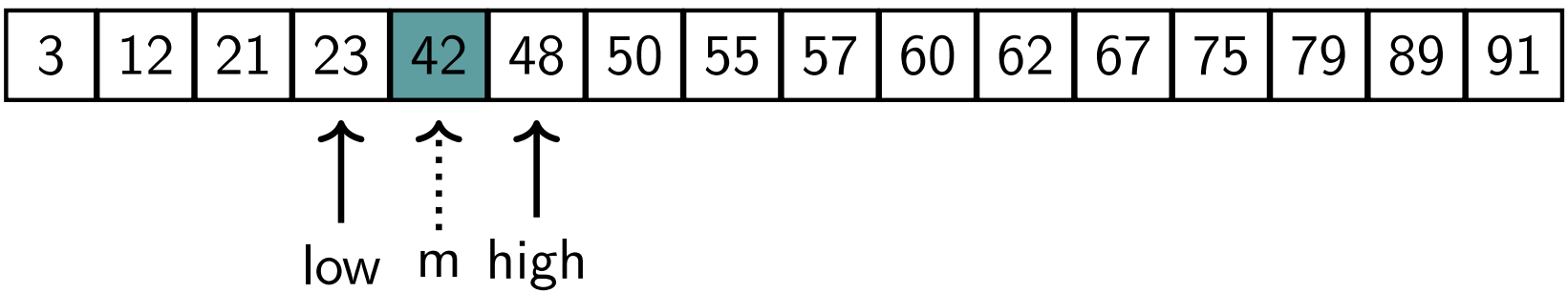

The bug that “fixed itself”

```
while (days > 365) {
  if (IsLeapYear(year)) {
    if (days > 366) {
      days -= 366;
      year += 1;
    }
  } else {
    days -= 365;
    year += 1;
  }
}
```

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Perhaps beginner's glitches only . . . ?

Proving that Android's, Java's and Python's sorting algorithm is broken (and showing how to fix it)

February 24, 2015 Envisage Written by Stijn de Gouw. $s

Tim Peters developed the Timsort hybrid sorting algorithm in 2002. It is a clever combination of ideas from merge sort and insertion sort, and designed to perform well on real world data. TimSort was first developed for Python, but later ported to Java (where it appears as java.util.Collections.sort and java.util.Arrays.sort) by Joshua Bloch (the designer of Java Collections who also pointed out that most binary search algorithms were broken). TimSort is today used as the default sorting algorithm for Android SDK, Sun's JDK and OpenJDK. Given the popularity of these platforms this means that the number of computers, cloud services and mobile phones that use TimSort for sorting is well into the billions.

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And as consequence?

- hands off recursion?
- base cases (and special cases) are particular *error prone* ("one-off" errors)
- working in most cases $\neq$ correct

> "Program testing can be used to show the presence of bugs, but never to show their absence"
>
> Dijkstra 1970, p. 7

- ultimately: careful reasoning needed ("correctness proof") angesagt.

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Further issues

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- in-direct recursion ("call-backs")
- induction & recursion
- inductive/rekursive data structures and corresponding algorithms (e.g. trees)
- complexity
- ...

Kudos

Martin Steffen

The slides where done with

- gnu emacs *org-mode* ("gnu's not Unix")
- LaTeX, und
- TikZ ("TikZ ist kein Zeichenprogram")

The "design" owes inspiration the elaborate style-files of the Uni Lübeck (M. Leucker, V. Stolz).

Sources

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Most of the material is "common knowledge" and I did not base the lecture on any specific book or source. Similar examples can be found in basically all introductions to Java, or other programming languages, for that matter. Pictures, if not self-made graphics, are likewise from "creative commons". Particular internet finds might be clickable via embedded links.