Higher Order Subtyping

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Overview

- Introduction
- $F^{\omega}_{\leq}$
- Subtyping
- Typing

Introduction

$$
\begin{tikzcd}
 & F \arrow[dl] \arrow[dr] & \\
F_{\leq} \arrow[dr] & & F^{\omega} \arrow[dl] \\
 & F^{\omega}_{\leq} &
\end{tikzcd}
$$

- F second order polymorphic λ-calculus [Girard, Reynolds]
- $F_{\leq}$ F + subtyping [Cardelli, Wegner]
- F^{ω} higher order polymorphism [Girard]
- $F^{\omega}_{\leq}$ higher order subtyping [Cardelli, Mitchell]

F

polymorphism by

types as function arguments

Example

reverse : $All(T)\ List(T) \to List(T)$

reverse Nat : $List(Nat) \to List(Nat)$

$\boxed{\text{Subtyping}}$

- Ordering on the types $\quad S \leqslant T$
- Subsumption

  If $t : S$ and $S \leqslant T$
  then $t : T$

$$\boxed{F_{\leq}}$$

- [Cardelli, Wegner]
- Extension of F with
  - subtyping
  - bounded quantification

Example:

listmax : All($T \leq$ Int) List(T) $\to T$

$F_{\leq}^{\omega}$

- [Cardelli, Mitchell]
- Extension of $F_{\leq}$ with functions from types to types
- models: [Cardelli, Longo]

$$\boxed{F^{\omega}_{\leq}}$$

terms: $t ::= x \mid \mathrm{fun}(x:T)\,t \mid t\;t$
$\mid \mathrm{fun}(A \leq T)\,t \mid t\,T$

types: $T ::= A \mid T \to T$
$\mid \mathrm{All}(A \leq T)\,T \mid \mathrm{Top}(K)$
$\mid \mathrm{Fun}(A:K)\,T \mid T\,T$

kinds: $K ::= \mathrm{Type} \mid K \to K$

$F^{\omega}_{\leq}$ Subtyping

statements $\quad \Gamma \vdash S \leq T$

rules: expressing

- $\leq$ is a preorder
- $\leq$ includes conversion

$$\frac{S =_{\beta} T}{\Gamma \vdash S \leq T}$$

- subtyping behaviour for each type constructor

  - A — as defined in
  - $Top(K)$ — maximal in
  - $S_1 \to S_2$ — contra/covariantly
  - $All(A \leq S_1)S_2$ — covariantly in the body
  - $Fun(A:K)S$ — pointwise
  - $S_1 S_2$ — pointwise

Example List : Type → Type

List Nat : Type

(5 7) : List Nat

Reduction

$$(\text{Fun}(A{:}\text{Type})\, A \to A)\,\text{Nat} \longrightarrow_\beta (\text{Nat} \to \text{Nat})$$

$$(\longrightarrow_\beta \cup \longleftarrow_\beta)^* = \; =_\beta \qquad \text{Conversion}$$

$F^{\omega}_{\leq}$ Subtyping (2)

Var
$$\frac{}{\Gamma_1, A \leq T, \Gamma_2 \vdash A \leq T}$$

Top
$$\frac{}{\Gamma \vdash S \leq Top(K)}$$

Arrow
$$\frac{\Gamma \vdash T_1 \leq S_1 \qquad \Gamma \vdash S_2 \leq T_2}{\Gamma \vdash S_1 \to S_2 \leq T_1 \to T_2}$$

Abstr
$$\frac{\Gamma, A \leq Top(K) \vdash S \leq T}{\Gamma \vdash Fun(A:K)S \leq Fun(A:K)T}$$

App
$$\frac{\Gamma \vdash S \leq U}{\Gamma \vdash S\,U \leq T\,U}$$

$F^{\omega}_{\leq}$ · Subtyping (3)

(All) $$\frac{\Gamma, A \leq \mathcal{U} \vdash S \leq T}{\Gamma \vdash \mathrm{All}(A \leq \mathcal{U})S \leq \mathrm{All}(A \leq \mathcal{U})T}$$

$$\frac{\Gamma, A \leq T_1 \vdash S_2 \leq T_2}{\Gamma \vdash \mathrm{All}(A \leq S_1)S_2 \leq \mathrm{All}(A \leq T_1)T_2}$$

Goal:
Algorithm for $\Gamma \vdash S \leq T$

- goal-directed proof search
  - no guessing of metavariables
  - no backtracking
- termination

Transitivity

$$\frac{\Gamma \vdash S \leq U \qquad \Gamma \vdash U \leq T}{\Gamma \vdash S \leq T}$$

Example $\quad \Gamma = \ldots B \leq C \ldots A \leq B \ldots$

$$\dfrac{\dfrac{}{\Gamma \vdash A \leq B}\,\text{Var} \qquad \dfrac{}{\Gamma \vdash B \leq C}\,\text{Var}}{\Gamma \vdash A \leq C}\,\text{Trans}$$

Trans cannot totally be eliminated!

$$\frac{A \uparrow_{\Gamma} B \qquad \overline{\Gamma \vdash B \leq C}}{\Gamma \vdash A \leq C}$$

Transitivity (2)

Example $\Gamma = \dots F \leqslant \mathrm{Id}$ with

$\mathrm{Id} = \mathrm{Fun}(B:K)\,B$

$$\dfrac{\dfrac{\overline{\Gamma \vdash F \leqslant \mathrm{Id}}\ \text{Var}}{\Gamma \vdash FA \leqslant \mathrm{Id}\,A}\ \text{App} \qquad \dfrac{\mathrm{Id}\,A =_\beta A}{\Gamma \vdash \mathrm{Id}\,A \leqslant A}\ \text{conv}}{\Gamma \vdash FA \leqslant A}\ \text{Trans}$$

$$\dfrac{FA \Uparrow_\Gamma \mathrm{Id}\,A \qquad \dfrac{\mathrm{Id}\,A =_\beta A}{\Gamma \vdash \mathrm{Id}\,A \leqslant A}\ \text{conv}}{\Gamma \vdash FA \leqslant A}$$

Promotion

$$A\, S_1 \ldots S_n \;\Uparrow_\Gamma\; \Gamma(A)\, S_1 \ldots S_n$$

upper bound of A in Γ

Rule of Promotion:

$$\frac{S \Uparrow_\Gamma U \qquad \Gamma \vdash U \leq T}{\Gamma \vdash S \leq T}$$

The Algorithm

- add the rule of Promotion
- eliminate Trans by cut-elimination
- replace conversion by normalizing reduction

$\Downarrow$

$\boxed{\text{Algorithm}}$

- soundness
- completeness
- termination

Typing $\Gamma \vdash t : T$

- standard adaptation of techniques for $F_{\leq}$
- comparatively easy
- only source of nondeterminism:

$$\text{Subsumption} \quad \frac{\Gamma \vdash t : S \qquad \Gamma \vdash S \leq T}{\Gamma \vdash t : T}$$

- "Elimination" of this rule by analysis of minimal types

⇨ Algorithm for $\Gamma \vdash t : T$, where T is minimal for t in Γ

related work

[Compagnoni]: Subtyping in $F_\wedge^\omega$ is decidable

[Curien, Ghelli]: $F_\leq$

[Breazu-Tannen, Coquand, Gunter, Scedrov]: $F_\leq$