Polarized Higher-Order Subtyping

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Overview

- Motivation
- Types & object-orientation
- “F-omega-sub”
- Decidability
- Conclusion

Typed programming

- "semantical" phase of compilers:

lexer ⇝	parser ⇝	type system ⇝	"oper. semantics"
regular	context-free	?	undecidable

- strong type safety:

  well-typed programs are free of run-time errors

- preferably: statically checkable ($\Rightarrow$ efficiency)

Parametric polymorphism

- polymorphic: a program can carry more than one type
- Example: swapping arguments

$$swap(x{:}\mathbb{N}, y{:}\mathbb{B}) \quad = \quad (y, x) \quad : \quad \mathbb{N} \times \mathbb{B} \rightarrow \mathbb{B} \times \mathbb{N}$$

- preferable: one generic *swap*-function for all types

$$swap : \forall X, Y. X \times Y \rightarrow Y \times X$$
$$swap\ X\ Y\ (x{:}X, y{:}Y) = (y, x)$$

⇒ **parametric/universal** polymorphism

Subtyping

Is S a **subtype** of T, then a program of type S can be safely used in place where a program of type T is expected

⇒ order on the types ($\leq$)

- intuitively: subsets $S \subseteq T$, e.g. $Int \leq Real$.
- combination with universal polymorphism:

$$list\_max : \forall X \leq Ord.(List\ of\ X) \to X$$

- bounded universal quantification

Type operators

- Example 1: "$\boldsymbol{List\ of}$ _" is no type, only "$\boldsymbol{List\ of}\ \mathbb{N}$" is, e.g.

$$[4, 5, 0] : \boldsymbol{List\ of}\ \mathbb{N}$$

⇒ type **operator** = function from types to types

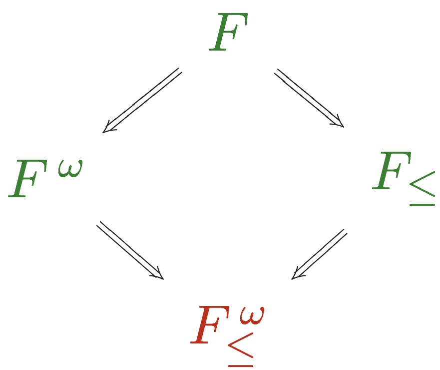
- Example 2: signature/method interface of objects

$$\begin{aligned} PointSig \quad of \;\; X = \{ \;\; & getx : X \rightarrow \mathbb{N}, \\ & setx : X \rightarrow \mathbb{N} \rightarrow X \; \} \end{aligned}$$

Features (cont.)

- polymorphism
  - universal polymorphism
  - subtyping
- higher-order functions
- encapsulation
- inheritance
- late/dynamic binding
- ...

Formal model: typed λ-calculi



- F: the polymorphic λ-calculus [Girard, 1971] [Reynolds, 1974]
- $F_{\leq}$: [Cardelli and Wegner, 1985] ...
- F^{ω} [Girard, 1971]
- $F^{\omega}_{\leq}$ [Cardelli, 1990] [Mitchell, 1990] ...

$F^{\omega}_{\leq}$ as OO-calculus

- [Hofmann and Pierce, 1995]: $F^{\omega}_{\leq}$ as base calculus for OO-languages
  - class-based
  - single inheritance
  - encapsulation (using $\exists$)

  provided:

  signature/method interface = monotone type operator

$$PointSig = Fun(X).\{\ getx : X \to \mathbb{N},\ setx : X \to \mathbb{N} \to \boldsymbol{X}\ \}$$

- class-inheritance $\Rightarrow$ subtype relation between instances
- absence of binary methods
- "inheritance of proofs" [Hofmann et al., 1998]

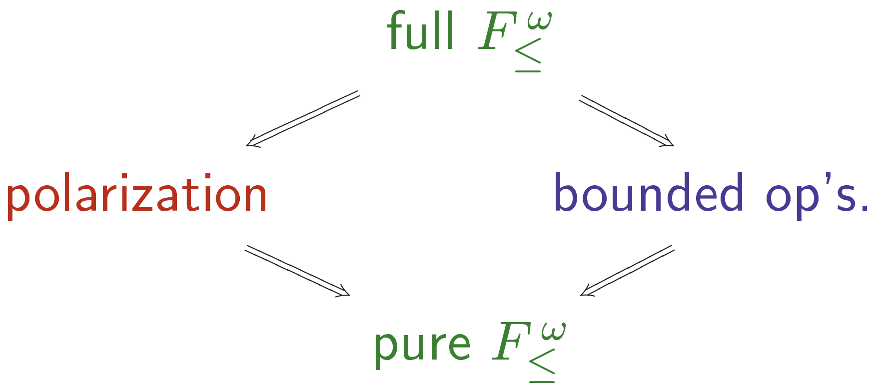
Example: more flexible typing

- Cf. [Duggan and Compagnoni, 1999] (for object type constructors)
- Example:

If $\quad Int \leq Real,$

then $\quad \boldsymbol{Array\ of}\ Int \leq \boldsymbol{Array\ of}\ Real\ ?$

$F^{\omega}_{\leq}$ = "the" calculus of higher-order subtyping?



- full $F^{\omega}_{\leq}$: [Cardelli, 1990]
- bounded operator abstraction: [Compagnoni and Goguen, 1997]

What's next

- fix the syntax
- axiomatize static properties
  - when does program t carries type T?
  - when is type S a subtype of type T?

$\Rightarrow$ formal deduction system

Syntax of $F^\omega_\leq$

- three levels: programs, types and kinds

t	$::=$	$x \mid fun(x{:}T)t \mid t\ t$	functions
	$\mid$	$fun(X \leq T)t \mid t\ T$	universal polymorphism, $\leq$
T	$::=$	$T \to T$	functions
	$\mid$	$All(X \leq T)T \mid X$	universal polymorphism, $\leq$
	$\mid$	$Top(K)$	
	$\mid$	$Fun(X{:}K)T \mid T\ T$	type operators
K	$::=$	$\star \mid K \to K$	kinds (= "type of types")

Judgments & rules

- So far: syntax only (context-free), but no relationships

⇒ Judgments (e.g.):

$\Gamma \vdash t : T$	program t is of type T
$\Gamma \vdash S \leq T$	S is a subtype of T
$\Gamma \vdash T : K$	type T is of kind K

- Dependency: Subsumption

$$\frac{\Gamma \vdash \boldsymbol{t : S} \qquad \Gamma \vdash \boldsymbol{S \leq T}}{\Gamma \vdash \boldsymbol{t : T}} \qquad (\textsc{Sub})$$

$F^{\omega}_{\leq}$: subtype system

- axiomatization of $\leq$ by deduction rules
- two classes of $\leq$-rules

1. language-independent properties of $\leq$, e.g. transitivity.

$$\frac{\Gamma \vdash \boldsymbol{S} \leq \boldsymbol{U} \quad \Gamma \vdash \boldsymbol{U} \leq \boldsymbol{T}}{\Gamma \vdash \boldsymbol{S} \leq \boldsymbol{T}} \tag{S-Trans}$$

2. structural, e.g. for $\rightarrow$–types

$$\frac{\Gamma \vdash \boldsymbol{T_1} \leq \boldsymbol{S_1} \qquad \Gamma \vdash \boldsymbol{S_2} \leq \boldsymbol{T_2}}{\Gamma \vdash \boldsymbol{S_1} \rightarrow \boldsymbol{S_2} \leq \boldsymbol{T_1} \rightarrow \boldsymbol{T_2}} \tag{S-Arrow}$$

$F^{\omega}_{\leq}$ + monotonicity

- extension of $F^{\omega}_{\leq}$ by monotonicity information ⇒

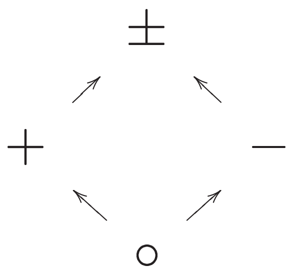
"polarized $F^{\omega}_{\leq}$"

If $\quad Int \leq Real,$

then $\quad \boldsymbol{List\ of}\ Int \leq \boldsymbol{List\ of}\ Real?$

$$\frac{\ldots \boldsymbol{X_1 \leq X_2} \ldots \vdash T\ X_1 \leq T\ X_2}{\Gamma \vdash T \in K_1 \rightarrow^{+} K_2}$$

- all in all: 4 polarities ⇒ subkinding



Goal

- Given: specification of the (sub–)type systems
- Needed: algorithm to check the judgments.

$$\text{type system} \quad \overset{?}{\Longrightarrow} \quad \text{type checker}$$

Where is the problem, then?

structural rules $\rightarrow$, $\forall$ ... straightforward (or almost ...)

$$\frac{\Gamma \vdash T_1 \leq S_1 \qquad \Gamma \vdash S_2 \leq T_2}{\Gamma \vdash S_1 \rightarrow S_2 \leq T_1 \rightarrow T_2}$$

but not

1. transitivity:

$$\frac{\Gamma \vdash S \leq U \quad \Gamma \vdash U \leq T}{\Gamma \vdash S \leq T}$$

2. conversion

$$\frac{S =_\beta S' \qquad \Gamma \vdash S' \leq T' \qquad T' =_\beta T}{\Gamma \vdash S \leq T}$$

Transitivity

- Goal: rule of transitivity is superfluous = "cut elimination"

$$\frac{U_1 \le S_1 \quad S_2 \le U_2}{S_1 \to S_2 \le U_1 \to U_2} + \frac{T_1 \le U_1 \quad U_2 \le S_2}{U_1 \to U_2 \le T_1 \to T_2}$$

$$\frac{\dfrac{T_1 \le U_1 \quad U_1 \le S_1}{T_1 \le S_1} \qquad \dfrac{S_2 \le U_2 \quad U_2 \le T_2}{S_2 \le T_2}}{S_1 \to S_2 \le T_1 \to T_2}$$

- Problems:
  - S-TRANS is not superfluous (known twist)
  - destroys the normal form

Eliminate cut?

- alas, $\textsc{S-Trans}$ is not superfluous:

- example:[1] assume $\Gamma = \ldots X \leq Y, Y \leq Z \ldots.$

$$\frac{\Gamma \vdash X \leq Y \qquad \Gamma \vdash Y \leq Z}{\Gamma \vdash X \leq Z}$$

- solution: add a new rule

$$\frac{\Gamma \vdash \Gamma(X) \leq T}{\Gamma \vdash X \leq T}$$

[1] one can have more complicated examples in F^{ω}

Conversion

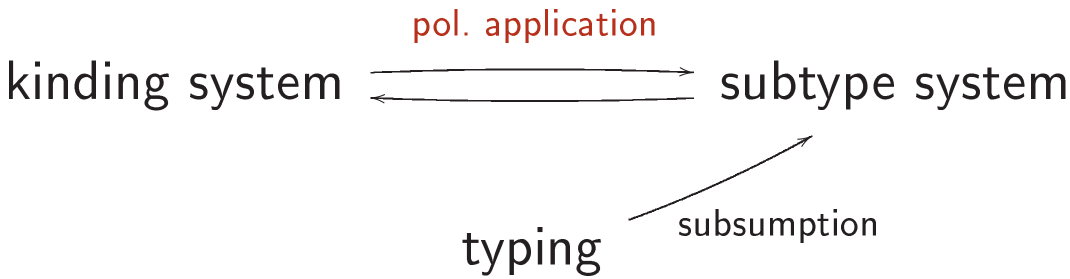
- Goal: Using normal forms only
- instead of undirected conversion: reduction

$$\frac{S =_\beta S' \qquad T =_\beta T' \qquad \Gamma \vdash S' \leq T'}{\Gamma \vdash S \leq T}$$

- For instance: for arrow-types

$$\frac{S \rightarrow^*_{\beta\top} S_1 \rightarrow S_2 \qquad T \rightarrow^*_{\beta\top} T_1 \rightarrow T_2 \qquad \Gamma \vdash T_1 \leq S_1 \in \star \qquad \Gamma \vdash S_2 \leq T_2 \in \star}{\Gamma \vdash S \leq T \in \star}$$

Additional problems



$$\frac{\begin{array}{c}\cdots\\ \ldots \boldsymbol{X_1} \leq \boldsymbol{X_2} \ldots \vdash T\ X_1 \leq T\ X_2\end{array}}{\Gamma \vdash T \in K_1 \rightarrow^{+} K_2}$$

- break the direct interdependence of subtyping and kinding $\Rightarrow$ "stratification"
- termination
- generalization of $\forall$-subtyping rule
- "antisymmetry" of $\leq$ (cf. [Compagnoni and Goguen, 1999])

Results

Theorem. *Subtyping* $\Gamma \vdash S \leq T : K$ *and kinding* $\Gamma \vdash T : K$ *for polarized* $F^{\omega}_{\leq}$ *are decidable.*

Proposition. *Every well-typed program has a minimal type.*

Corollary. *Typing* $\Gamma \vdash t : T$ *for polarized* $F^{\omega}_{\leq}$ *is decidable.*

Future work

- Model (for instance PER-model)
- decidability for the full calculus ([Compagnoni and Goguen, 1997]):

$$Fun(X \leq S)T \qquad \text{instead of} \qquad Fun(X : K)T$$

- local type inference (e.g. [Pierce and Turner, 1998] for $F_{\leq}$, in Pict)

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