## Classification of Programs

There are two classes of programs:

**Computational Programs:** Run in order to produce a final result on termination.

Can be modeled as a black box.


Specified in terms of Input/Output relations.

**Example:**

The program which computes

$$y = 1 + 3 + \cdots + (2x - 1)$$

Can be specified by the requirement

$$y = x^2.$$



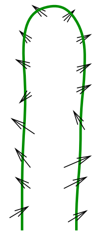
## Reactive Programs

Programs whose role is to maintain an ongoing interaction with their environments, rather than produce a final result upon termination.

**Examples:** Air traffic control system, Programs controlling mechanical devices such as a train, a plane, or ongoing processes such as a nuclear reactor.

Termination is not necessarily expected, and the important functionality is interaction with the environment.

Can be viewed as a green cactus (?)



Such programs must be specified and verified in terms of their behaviors.

## A Framework for Reactive Systems Verification

- A computational model providing an abstract syntactic base for all reactive systems. We use fair Discrete systems (FDS).
- A Specification Language for specifying systems and their properties. We use linear temporal logic (LTL).
- An Implementation Language for describing proposed implementations (both software and hardware). We use SPL, a simple programming language.
- Verification Techniques for validating that an implementation satisfies a specification. Practiced approaches:
  - Algorithmic verification methods for exploratory verification of finite-state systems: Enumerative and Symbolic variants.
  - A deductive methodology based on theorem-proving methods. Can accommodate infinite-state systems, but requires user interaction.



## Fair Discrete Systems

A fair discrete system (FDS) $\mathcal{D} = \langle V, \mathcal{O}, \Theta, \rho, \mathcal{J}, \mathcal{C}\rangle$ consists of:

- $V$ – A finite set of typed state variables. A $V$-state $s$ is an interpretation of $V$. $\Sigma_V$ – the set of all $V$-states.
- $\mathcal{O} \subseteq V$ – A set of observable variables.
- $\Theta$ – An initial condition. A satisfiable assertion that characterizes the initial states.
- $\rho$ – A transition relation. An assertion $\rho(V, V')$, referring to both unprimed (current) and primed (next) versions of the state variables. For example, $x' = x + 1$ corresponds to the assignment $x := x + 1$.
- $\mathcal{J} = \{J_1, \ldots, J_k\}$ A set of justice (weak fairness) requirements. Ensure that a computation has infinitely many $J_i$-states for each $J_i$, $i = 1, \ldots, k$.
- $\mathcal{C} = \{\langle p_1, q_1\rangle, \ldots \langle p_n, q_n\rangle\}$ A set of compassion (strong fairness) requirements. Infinitely many $p_i$-states imply infinitely many $q_i$-states.

## A Simple Programming Language: SPL

A language allowing composition of parallel processes communicating by shared variables as well as message passing.

## Example: Program ANY-Y

Consider the program

$$x,\ y: \textbf{natural initially } x = y = 0$$

$$\left[\begin{array}{l} \ell_0: \quad \textbf{while } x = 0 \textbf{ do} \\ \quad [\ell_1: \ y := y + 1] \\ \ell_2: \end{array}\right] \quad \| \quad \left[\begin{array}{l} m_0: \quad x := 1 \\ m_1: \end{array}\right]$$

$$- \quad P_1 \quad - \qquad\qquad - \quad P_2 \quad -$$



## The Corresponding FDS

- State Variables $V$: $\left(\begin{array}{lcl} x, y & : & \text{natural} \\ \pi_1 & : & \{\ell_0, \ell_1, \ell_2\} \\ \pi_2 & : & \{m_0, m_1\} \end{array}\right)$.
- Initial condition: $\Theta: \ \pi_1 = \ell_0 \ \wedge \ \pi_2 = m_0 \ \wedge \ x = y = 0$.
- Transition Relation: $\rho$: $\rho_I \vee \rho_{\ell_0} \vee \rho_{\ell_1} \vee \rho_{m_0}$, with appropriate disjunct for each statement. For example, the disjuncts $\rho_I$ and $\rho_{\ell_0}$ are

$$\rho_I: \quad \pi_1' = \pi_1 \ \wedge \ \pi_2' = \pi_2 \ \wedge \ x' = x \ \wedge \ y' = y$$

$$\rho_{\ell_0}: \quad \pi_1 = \ell_0 \quad \wedge \quad \left(\begin{array}{c} x = 0 \ \wedge \ \pi_1' = \ell_1 \\ \vee \\ x \neq 0 \ \wedge \ \pi_1' = \ell_2 \end{array}\right)$$

$$\wedge \quad \pi_2' = \pi_2 \ \wedge \ x' = x \ \wedge \ y' = y$$

- Justice set: $\mathcal{J}$: $\{\neg at\_\ell_0, \neg at\_\ell_1, \neg at\_m_0\}$.
- Compassion set: $\mathcal{C}$: $\emptyset$.

## Computations

Let $\mathcal{D}$ be an FDS for which the above components have been identified. The state $s'$ is defined to be a $\mathcal{D}$-successor of state $s$ if

$$\langle s, s' \rangle \models \rho_{\mathcal{D}}(V, V').$$

We define a computation of $\mathcal{D}$ to be an infinite sequence of states

$$\sigma : s_0, s_1, s_2, \ldots,$$

satisfying the following requirements:

- Initiality: $s_0$ is initial, i.e., $s_0 \models \Theta$.
- Consecution: For each $j = 0, 1, \ldots$, state $s_{j+1}$ is a $\mathcal{D}$-successor of state $s_j$.
- Justice: For each $J \in \mathcal{J}$, $\sigma$ contains infinitely many $J$-positions
- Compassion: For each $\langle p, q \rangle \in \mathcal{C}$, if $\sigma$ contains infinitely many $p$-positions, it must also contain infinitely many $q$-positions.



## Examples of Computations

Identification of the FDS $\mathcal{D}_P$ corresponding to a program $P$ gives rise to a set of computations $\mathcal{C}omp(P) = \mathcal{C}omp(\mathcal{D}_P)$.

The following computation of program ANY-Y ccorresponds to the case that $m_0$ is the first executed statement:

$$\langle \pi_1\colon \ell_0,\ \pi_2\colon m_0\ ;\ x\colon 0,\ y\colon 0\rangle \xrightarrow{m_0} \langle \pi_1\colon \ell_0,\ \pi_2\colon m_1\ ;\ x\colon 1,\ y\colon 0\rangle \xrightarrow{\ell_0}$$
$$\langle \pi_1\colon \ell_2,\ \pi_2\colon m_1\ ;\ x\colon 1,\ y\colon 0\rangle \xrightarrow{\tau_I} \cdots \xrightarrow{\tau_I} \cdots$$

The following computation corresponds to the case that statement $\ell_1$ is executed before $m_0$.

$$\langle \pi_1\colon \ell_0,\ \pi_2\colon m_0\ ;\ x\colon 0,\ y\colon 0\rangle \xrightarrow{\ell_0} \langle \pi_1\colon \ell_1,\ \pi_2\colon m_0\ ;\ x\colon 0,\ y\colon 0\rangle \xrightarrow{\ell_1}$$
$$\langle \pi_1\colon \ell_0,\ \pi_2\colon m_0\ ;\ x\colon 0,\ y\colon 1\rangle \xrightarrow{m_0} \langle \pi_1\colon \ell_0,\ \pi_2\colon m_1\ ;\ x\colon 1,\ y\colon 1\rangle \xrightarrow{\ell_0}$$
$$\langle \pi_1\colon \ell_2,\ \pi_2\colon m_1\ ;\ x\colon 1,\ y\colon 1\rangle \xrightarrow{\tau_I} \cdots \xrightarrow{\tau_I} \cdots$$

In a similar way, we can construct for each $n \geq 0$ a computation that executes the body of statement $\ell_0$ $n$ times and then terminates in the final state

$$\langle \pi_1\colon \ell_2,\ \pi_2\colon m_1\ ;\ x\colon 1,\ y\colon n\rangle.$$

## A Non-Computation

While we can delay termination of the program for an arbitrary long time, we cannot postpone it forever.

Thus, the sequence

$$\langle \pi_1\colon \ell_0 ,\ \pi_2\colon m_0\ ;\ x\colon 0 ,\ y\colon 0\rangle \xrightarrow{\ell_0} \langle \pi_1\colon \ell_1 ,\ \pi_2\colon m_0\ ;\ x\colon 0 ,\ y\colon 0\rangle \xrightarrow{\ell_1}$$
$$\langle \pi_1\colon \ell_0 ,\ \pi_2\colon m_0\ ;\ x\colon 0 ,\ y\colon 1\rangle \xrightarrow{\ell_0} \langle \pi_1\colon \ell_1 ,\ \pi_2\colon m_0\ ;\ x\colon 0 ,\ y\colon 1\rangle \xrightarrow{\ell_1}$$
$$\langle \pi_1\colon \ell_0 ,\ \pi_2\colon m_0\ ;\ x\colon 0 ,\ y\colon 2\rangle \xrightarrow{\ell_0} \langle \pi_1\colon \ell_1 ,\ \pi_2\colon m_0\ ;\ x\colon 0 ,\ y\colon 2\rangle \xrightarrow{\ell_1}$$
$$\langle \pi_1\colon \ell_0 ,\ \pi_2\colon m_0\ ;\ x\colon 0 ,\ y\colon 3\rangle \xrightarrow{\ell_0} \cdots$$

in which statement $m_0$ is never executed is not an admissible computation. This is because it violates the justice requirement $\neg at\_m_0$ contributed by statement $m_0$, by having no states in which this requirement holds.

This illustrates how the requirement of justice ensures that program ANY-Y always terminates.

Justice guarantees that every (enabled) process eventually progresses, in spite of the representation of concurrency by interleaving.



## SPL: Syntax

### Statements

- **skip** – A do-nothing statement.
- $y := e$ – an assignment. Assign the value of expression $e$ to variable $y$.
- **await** $b$ – Wait until the value of the boolean expression $b$ becomes true.
- Compound Statements – If $b$ is a boolean expression, and $S$, $S_1$, $S_2$ are statements, then so are
  - $S_1; S_2$ – Concatenation. Execute $S_1$ first and then $S_2$.
  - $[S]$ – Grouping.
  - **if** $b$ **then** $S_1$ **else** $S_2$ – Conditional. Execute $S_1$ if $b$ evaluates to $1$ (true). Otherwise, execute $S_2$.
  - **while** $b$ **do** $S$ – a while statement. Repeatedly execute $S$ as long as $b$ evaluates to $1$. If initially $b \sim 0$ then this is equivalent to **skip**.
- Abbreviations
  - **if** $b$ **then** $S$ $\sim$ **if** $b$ **then** $S$ **else skip**
  - **when** $b$ **do** $S$ $\sim$ [**await** $b$; $S$]

## Syntax – Declaration

A declaration has the form

$\{\langle mode \rangle\}$ variable$_1$, variable$_2$,. . .,variable$_k$: $\langle type \rangle$ {where $\varphi$}

where the optional $\langle mode \rangle$ is one of the following:

- **in** – Specifies variables that are input to the program/process. Cannot be modified inside the unit.
- **local** – Specifies variables that are local to the program/process but are not recognized out of it.
- **out** – Variables that are an output of the program/process. Cannot be modified outside the unit.
- **in-out** – Variables which can be modified both inside and outside the unit.

The $\langle type \rangle$ can be a basic type which are **integer**, **natural**, **bool** (boolean) or $[L..U]$ (an integer in the range $L..U$).

It can also be an array type of the form **array** $[L..U]$ **of** $\langle type \rangle$.

The optional where clause specifies constraints on the initial values of variables.



## Syntax – Processes and Programs

A process has the form

$$\{\langle process\_name \rangle :: \}\ [\ \{\langle declarations \rangle ; \}\ \langle statement \rangle ; \langle label \rangle\ : ]$$

where $\langle declarations \rangle$ are $0$ or more declarations, separated by ";". Thus, every process terminates in a label which denotes the location of control after the process has terminated. We refer to the statement as the body of the process.

A program has the form

$$\{\langle declarations \rangle ; \}\ P_1 \| \cdots \| P_k,$$

where each $P_i$, $i = 1, \ldots, k$ is a process.

# Labels

It is assumed that every statement is labeled. For a statement $S$, we define $pre(S)$ to be the preceding label which is closest to $S$ in the program.

We also define $post(S)$ inductively as follows:

- If $S; \ell :$ is the body of a process, then $post(S) = \ell$.
- If $S = [S_1; \cdots S_k]$, then $post(S_i) = pre(S_{i+1})$ for $i = 1, \ldots, k-1$ and $post(S_k) = post(S)$.
- If $S =$ **if** $b$ **then** $S_1$ **else** $S_2$ then $post(S_1) = post(S_2) = post(S)$.
- If $S =$ **while** $b$ **do** $S_1$ then $post(S_1) = pre(S)$.

For a label $\ell_i$ within process $P_j$, we write $at\_\ell_i$ as an abbreviation for

$$\pi_j = \ell_i$$



# For Example

Consider the following process:

$$P_1 :: \left[\begin{array}{ll} \ell_0 : & x := 1 \\ \ell_1 : & \textbf{while } y > 0 \textbf{ do} \\ & \quad \left[\begin{array}{ll} \ell_2 : & x := x + 2 \\ \ell_3 : & y := y - 1 \end{array}\right] \\ \ell_4 : & x := x - 1 \\ \ell_5 : & \end{array}\right]$$

Then we have:

| $S$ | $post(S)$ |
|---|---|
| $\ell_0 : \cdots;\ \ell_1 : \cdots;\ \ell_4$ | $\ell_5$ |
| $\ell_0 : \cdots$ | $\ell_1$ |
| $\ell_1 : \cdots$ | $\ell_4$ |
| $\ell_4 : \cdots$ | $\ell_5$ |
| $\ell_2 : \cdots;\ \ell_3 : \cdots$ | $\ell_1$ |
| $\ell_2 : \cdots$ | $\ell_3$ |
| $\ell_3 : \cdots$ | $\ell_1$ |

## SPL: Semantics

Let $P ::\ \ declaration;\ P_1 \| \cdots \| P_k$ be a program. We proceed to construct the FDS $\mathcal{D}_P$ corresponding to program $P$.

**State Variables** As the state variables, we take all the variables declared in the program and add to them a set of control variables

$\pi_1, \ldots, \pi_k$

For each $i = 1, \ldots, k$, the domain of $\pi_i$ is the set of labels appearing in process $S_i$.

For example, for program ANY-Y, the state variables are

$$V: \begin{pmatrix} x, y & : & \text{natural} \\ \pi_1 & : & \{\ell_0, \ell_1, \ell_2\} \\ \pi_2 & : & \{m_0, m_1\} \end{pmatrix}$$

**Observable Variables** At this point, we take $\mathcal{O} = V$.

**Initial Condition** As the initial condition, we take the conjunction of all the **where** clauses plus the conjunction

$\pi_1 = pre(S_1)\ \wedge\ \cdots\ \wedge\ \pi_k = pre(S_k)$

For example, the initial condition for program ANY-Y is given by

$$\Theta:\ \pi_1 = \ell_0\ \wedge\ \pi_2 = m_0\ \wedge\ x = y = 0.$$



## The Transition Relation

For a subset of variables $U \subseteq V$, we denote $pres(U) = \bigwedge_{x \in U} (x' = x)$.

The transition relation $\rho$ is formed as a disjunction which standardly contains the disjunct $\rho_{idle} : pres(V)$. In addition, each statement $S$ in the program, excluding concatenation statements, contributes a disjunct $\rho_S$ according to the following recipe:

- The statement $S = \textbf{skip}$ in process $P_i$ contributes the disjunct

$$\pi_i = pre(S)\ \wedge\ \pi_i' = post(S)\ \wedge\ pres(V - \{\pi_i\})$$

- The statement $S = [y := e]$ in process $P_i$ contributes the disjunct

$$\pi_i = pre(S)\ \wedge\ \pi_i' = post(S)\ \wedge\ y' = e\ \wedge\ pres(V - \{\pi_i, y\})$$

For example, statement $\ell_1$ in program ANY-Y contributes the disjunct

$$\pi_1 = \ell_1\ \wedge\ \pi_1' = \ell_0\ \wedge\ y' = y + 1\ \wedge\ pres(\{\pi_2, x\})$$

## Transition Relation – Continued

- The statement $S$ = **await** $b$ in process $P_i$ contributes the disjunct

$$\pi_i = pre(S) \;\wedge\; b \;\wedge\; \pi_i' = post(S) \;\wedge\; pres(V - \{\pi_i\})$$

- The statement $S$ = **if** $b$ **then** $S_1$ **else** $S_2$ in process $P_i$ contributes the disjunct

$$\pi_i = pre(S) \;\wedge\; \left(\begin{array}{ccc} & b & \wedge & \pi_i' = pre(S_1) \\ \vee & \neg b & \wedge & \pi_i' = pre(S_2)\end{array}\right) \;\wedge\; pres(V - \{\pi_i\})$$

- The statement $S$ = **while** $b$ **do** $S_1$ in process $P_i$ contributes the disjunct

$$\pi_i = pre(S) \;\wedge\; \left(\begin{array}{ccc} & b & \wedge & \pi_i' = pre(S_1) \\ \vee & \neg b & \wedge & \pi_i' = post(S)\end{array}\right) \;\wedge\; pres(V - \{\pi_i\})$$

For example, statement $\ell_0$ of program ANY-Y contributes the disjunct

$$\pi_1 = \ell_0 \;\wedge\; \left(\begin{array}{ccc} & x = 0 & \wedge & \pi_1' = \ell_1 \\ \vee & x \neq 0 & \wedge & \pi_1' = \ell_2\end{array}\right) \;\wedge\; pres(\{\pi_2, x, y\})$$



## Justice Requirements

Each occurrence within process $P_i$ of a statement $S$ which is a **skip**, an assignment, a conditional or a while statement, contributes to the justice set the requirement

$$J_S: \quad \pi_i \neq pre(S)$$

An occurrence within $P_i$ of a statement $S$ = **await** $b$, contributes the justice requirement:

$$J_S: \quad \neg(\pi_i = pre(S) \;\wedge\; b).$$

For example, the justice set for program ANY-Y is

$$\mathcal{J}: \quad \{\pi_1 \neq \ell_0, \;\; \pi_1 \neq \ell_1, \;\; \pi_2 \neq m_0\}$$

The implication of the justice requirements are:

No statement is continuously enabled without being executed.

or, equivalently,

If $S$ is continuously enabled it must eventually be executed.

## Justice is not Enough. You also Need Compassion

The following program MUX-SEM, implements mutual exclusion by semaphores.

$$y: \textbf{natural initially } y = 1$$

$$\left[\begin{array}{l} \ell_0: \ \textbf{loop forever do} \\ \left[\begin{array}{ll} \ell_1: & \textbf{Non-critical} \\ \ell_2: & \textbf{request } y \\ \ell_3: & \textbf{Critical} \\ \ell_4: & \textbf{release } y \end{array}\right] \end{array}\right] \ \Big\| \ \left[\begin{array}{l} m_0: \ \textbf{loop forever do} \\ \left[\begin{array}{ll} m_1: & \textbf{Non-critical} \\ m_2: & \textbf{request } y \\ m_3: & \textbf{Critical} \\ m_4: & \textbf{release } y \end{array}\right] \end{array}\right]$$

$$- \ P_1 \ - \qquad\qquad\qquad - \ P_2 \ -$$

The semaphore instructions **request** $y$ and **release** $y$ respectively stand for

$$\langle \textbf{await } y > 0;\ y := y - 1 \rangle \quad \text{and} \quad y := y + 1.$$

The compassion set of this program consists of

$$\mathcal{C}: \quad \{(at\_\ell_2 \wedge y > 0, \quad at\_\ell_3), \quad (at\_m_2 \wedge y > 0, \quad at\_m_3)\}.$$



## Program MUX-SEM

should satisfy the following two requirements:

- Mutual Exclusion – No computation of the program can include a state in which process $P_1$ is at $\ell_3$ while $P_2$ is at $m_3$.
- Accessibility – Whenever process $P_1$ is at $\ell_2$, it shall eventually reach it's critical section at $\ell_3$. Similar requirement for $P_2$.

Consider the state sequence:

$$\begin{array}{lllll} \sigma: & \langle \ell_0,\ m_0,\ 1\rangle \longrightarrow & \cdots \longrightarrow & \boxed{\langle \ell_2,\ m_2,\ 1\rangle} \xrightarrow{m_2} \\ & \boxed{\langle \ell_2,\ m_3,\ 0\rangle} \xrightarrow{m_3} & \langle \ell_2,\ m_4,\ 0\rangle \xrightarrow{m_4} & \\ & \langle \ell_2,\ m_0,\ 1\rangle \xrightarrow{m_0} & \langle \ell_2,\ m_1,\ 1\rangle \xrightarrow{m_1} & \boxed{\langle \ell_2,\ m_2,\ 1\rangle} \xrightarrow{m_2} \\ & \boxed{\langle \ell_2,\ m_3,\ 0\rangle} \longrightarrow & \cdots, & \end{array}$$

which violates accessibility for process $P_1$. We should not allow this state sequence as a computation.

If the only fairness requirement associated with statement $\ell_2$ : **request** $y$ were that of justice, the above state sequence would be a computation. This is because statement $\ell_2$ is not continuously enabled. In fact, it is disabled on all states of the form $\langle \ell_2,\ m_3,\ 0\rangle$.

# Compassion Saves the Day

Instead, we associate with statement $\ell_2$ : **request** $y$ the compassion requirement

$$(at\_\ell_2 \;\wedge\; y > 0, \quad at\_\ell_3)$$

implying

Statement $\ell_2$ cannot be infinitely often enabled without being executed

Due to this compassion requirement for $\ell_2$, the violating state sequence is not a computation, and accessibility is guaranteed.

**Conclusion:** Justice alone is not sufficient !!!



# Compassion Requirements

Each occurrence within $P_i$ of a statement $S$ = **request** $y$, contributes the compassion requirement:

$$C_S : \quad (\pi_i = pre(S) \;\wedge\; y > 0, \quad \pi_i = post(S)).$$