Notations:

An ω-sequence (or ω-word) over A is a function from ω into A. We also use the notation $(a_i)_{i\in\omega}$ to denote the ω-sequence a such that $\forall i \in \omega.\ a(i) = a_i$. A finite sequence w (or finite word) over A is a function from $\{0, \cdots, n-1\}$ into A; n is the length of w. We use the notation $a_0 \cdots a_{n-1}$ to denote the finite sequence w of length n such that $\forall i \in \{0, \cdots, n-1\}.\ w(i) = a_i$. Given an ω-sequence $\alpha = (a_i)_{i\in\omega}$, given $i, j \in \omega$, $\alpha(i, j)$ denotes the finite sequence $a_i \cdots a_{j-1}$. Notice that if $j \leq i$ then $\alpha(i, j)$ is the empty sequence which we denote by ϵ.

Chapter 3

Operational Specification

Usually also other properties than invariance properties are of interest when protocols and reactive systems are considered. For instance, if a communication protocol is considered then it is important to know that every message is eventually delivered.

Automata accepting ω-words, so-called ω-automata, represent a powerful tool for specifying a large class of systems and their properties.

3.1 Büchi Automata

Definition 3.1 [Büchi Automata]
A Büchi automaton $\mathcal{B}$ *over alphabet* A *is given by a quadrupel* $(Q, q_0, \longrightarrow, F)$*, where*

- Q *is a finite set of control locations.*
- $q_0 \in Q$ *is the initial control location.*
- $\longrightarrow \subseteq Q \times A \times Q$ *is the transition relation of* $\mathcal{B}$*. We write* $q \xrightarrow{a} q'$ *instead of* $(q, a, q') \in \longrightarrow$.
- $F \subseteq Q$ *is the set of* accepting *control locations.*

◇

Definition 3.2

- *A* run ρ *of* $\mathcal{B} = (Q, q_0, \longrightarrow, F)$ *over the* ω*-word* α *is an* ω*-sequence in* Q^ω *such that* $\rho(0) = q_0$ *and* $\forall i \in \omega \cdot \rho(i) \xrightarrow{\alpha(i)} \rho(i+1)$.
- *A run* ρ *is called* accepting*, if* $\mathit{inf}(\rho) \cap F \neq \emptyset$*, where* $\mathit{inf}(\rho) = \{q \in Q \mid \exists^\omega i \cdot \rho(i) = q\}$.

- $\alpha \in A^\omega$ *is* accepted *by* $\mathcal{B}$*, if there exists an accepting run of* $\mathcal{B}$ *over* α*.*

- *The language of* $\mathcal{B}$*, denoted by* $\mathcal{L}(\mathcal{B})$ *is the set of all* ω*-words accepted by* $\mathcal{B}$*.*

- *An* ω*-language* $L \subseteq A^\omega$ *is* recognized *by* $\mathcal{B}$*, if* $L = \mathcal{L}(\mathcal{B})$*; it is* Büchi recognizable*, if there exists a Büchi automaton that recognizes it.*

◇

Definition 3.3 *Given a Büchi automaton* $\mathcal{B} = (Q, q_0, \longrightarrow, F)$*,* $q, q' \in Q$*, and* $w \in A^*$*. Define* $q \xrightarrow{w} q'$ *and* $W_{qq'}$ *as follows:*

- $q \xrightarrow{w} q'$ *iff* $\exists l_0, \cdots, l_n \in Q$ *such that* $n = |w|$, $l_0 = q$, $l_n = q'$*, and* $\forall i \in \{0, \cdots, n-1\} \cdot l_i \xrightarrow{w(i)} l_{i+1}$.

- $W_{qq'} = \{w \in A^* \mid q \xrightarrow{w} q'\}$.

◇

Lemma 3.1 Let $\mathcal{B}$ be a Büchi automaton. Then,

$$\mathcal{L}(\mathcal{B}) = \bigcup_{q \in F} W_{q_0 q} W_{qq}^\omega.$$

□

Proof:

$$\begin{aligned}
\alpha \in \mathcal{L}(\mathcal{B}) &\iff \exists \text{ an accepting run over } \alpha \\
&\iff \exists \rho \in Q^\omega \cdot \exists q \in F \cdot \rho(0) = q_0 \wedge \\
&\qquad \exists (j_i)_{i \in \omega} \in \omega^\omega \cdot \forall i \in \omega \cdot (j_{i+1} > j_i \wedge \rho(j_i) = q \wedge q_0 \xrightarrow{\alpha(0, j_0)} q \wedge q \xrightarrow{\alpha(j_i, j_{i+1})} q) \\
&\iff \exists q \in F \cdot \exists (j_i)_{i \in \omega} \in \omega^\omega \cdot (\forall i \in \omega \cdot j_{i+1} > j_i) \wedge \alpha(0, j_0) \in W_{q_0 q} \wedge \\
&\qquad \forall i \in \omega \cdot \alpha(j_i, j_{i+1}) \in W_{qq} \\
&\iff \alpha \in \bigcup_{q \in F} W_{q_0 q} W_{qq}^\omega
\end{aligned}$$

■

Definition 3.4 *A word* $\alpha \in A^\omega$ *is called* ultimately periodic*, if there exists an increasing* ω*-sequence* $(j_i)_{i \in \omega} \in \omega^\omega$ *such that* $\forall i \in \omega \cdot \alpha(j_i, j_{i+1}) = \alpha(j_{i+1}, j_{i+2})$*, in other words there exist* $u, v \in A^*$ *such that* $\alpha = uv^\omega$*.* ◇

Proposition 3.2

1.) Let $\mathcal{B}$ be a Büchi automaton. Then, $\mathcal{L}(\mathcal{B}) \neq \emptyset$ iff $\mathcal{L}(\mathcal{B})$ contains an ultimately periodic word.

2.) The emptiness problem for Büchi automata is decidable.

□

Proof:

1.) The implication from right to left is immediate. Thus, let us consider the other one. Since $\mathcal{L}(\mathcal{B}) = \bigcup_{q \in F} W_{q_0 q} W_{qq}^{\omega}$, if $\mathcal{L}(\mathcal{B}) \neq \emptyset$ then there exists $q \in F$ such that $W_{q_0 q} \neq \emptyset$ and $W_{qq} \neq \emptyset$. Let $u \in W_{q_0 q}$ and $v \in W_{qq}$, then $uv^{\omega} \in \mathcal{L}(\mathcal{B})$.

2.) Follows from the fact that $W_{qq'}$ is a regular language in A^*.

■

Lemma 3.3

1.) If $V \subseteq A^*$ is regular then V^{ω} is Büchi recognizable.

2.) If $V \subseteq A^*$ is regular and $L \subseteq A^{\omega}$ is Büchi recognizable then $V \cdot L$ is also Büchi recognizable.

3.) If L_1 and L_2 are Büchi recognizable then $L_1 \cup L_2$ and $L_1 \cap L_2$ are also Büchi recognizable.

□

Proof:

1.) Since for every regular language V given by an automaton $\mathcal{A}$ we can construct an automaton $\mathcal{A}'$ such that no transition in $\mathcal{A}'$ leads to the initial control state, $\epsilon \notin \mathcal{L}(\mathcal{A}')$, and $\mathcal{L}(\mathcal{A}')^{\omega} = V^{\omega}$, we can assume without loss of generality that V is given by an automaton $\mathcal{A} = (Q, q_0, \longrightarrow, F)$ such that $\forall q \in Q \cdot q \not\longrightarrow q_0$ and $q_0 \notin F$.

Let $\mathcal{B} = (Q \setminus F, q_0, \longrightarrow_{\mathcal{B}}, \{q_0\})$ such that for all $q, q' \in Q \setminus F$ and $a \in A$

$$q \xrightarrow{a}_{\mathcal{B}} q' \iff ((q \xrightarrow{a} q' \wedge q' \notin F) \vee \exists q'' \in F \cdot q \xrightarrow{a} q'' \wedge q' = q_0).$$

We prove $\mathcal{L}(\mathcal{B}) \subseteq V^{\omega}$ and leave $V^{\omega} \subseteq \mathcal{L}(\mathcal{B})$ as exercise.

Let $\alpha \in \mathcal{L}(\mathcal{B})$. Then, there exists a run ρ over α and an increasing sequence $(j_i) \in \omega^{\omega}$ with $j_0 = 0$ and $\forall i \in \omega \cdot \rho(j_i) = q_0 \wedge \forall k \in \{j_i + 1, \cdots, j_{i+1} - 1\}.\ \rho(k) \neq q_0$. We prove $\alpha(j_i, j_{i+1}) \in V$, for every $i \in \omega$.

Thus, let $i \in \omega$. Since ρ is a run of $\mathcal{B}$, we have $\forall k \in \{j_i, \cdots, j_{i+1}-1\} \cdot \rho(k) \xrightarrow{\alpha(k)}_{\mathcal{B}} \rho(k+1)$. By the assumption on $\mathcal{A}$ and the definition of $\mathcal{B}$, $\forall k \in \{j_i, \cdots, j_{i+1}-2\} \cdot \rho(k) \xrightarrow{\alpha(k)} \rho(k+1)$ and $\exists q \in F \cdot \rho(j_{i+1}-1) \xrightarrow{\alpha(j_{i+1})} q$. Hence, there exists $q \in F$ such that $\rho(j_i), \cdots, \rho(j_{i+1})$ is a run of $\mathcal{A}$ over $\alpha(j_i, j_{i+1})$ and $\rho(j_{i+1}) = q$. This proves $\alpha(j_i, j_{i+1}) \in V$.

2.) Exercise.

3.) W.l.g. assume that L_1 and L_2 are given by $\mathcal{B}_i = (Q_i, q_{0i}, \longrightarrow_i, F_i)$, for $i = 1, 2$, with $Q_1 \cap Q_2 = \emptyset$.

Let $\mathcal{B} = (Q_1 \cup Q_2 \cup \{q_0\}, q_0, \longrightarrow, F_1 \cup F_2)$, where $q_0 \notin Q_1 \cup Q_2$ and $q \xrightarrow{a} q'$ iff either $q \xrightarrow{a}_1 q'$, or $q \xrightarrow{a}_2 q'$, or $q = q_0$ and $q_{01} \xrightarrow{a}_1 q'$ or $q_{02} \xrightarrow{a}_2 q'$. Then, it is not difficult to check that $\mathcal{L}(\mathcal{B}) = L_1 \cup L_2$.

Next, we construct a Büchi automaton $\mathcal{B}$ such that $\mathcal{L}(\mathcal{B}) = L_1 \cap L_2$.

Let $\mathcal{B} = (Q_1 \times Q_2 \times \{0,1,2\}, (q_{01}, q_{02}, 0), \longrightarrow, F)$, where $F = \{(q_1, q_2, 2) \mid q_1 \in Q_1, q_2 \in Q_2\}$ and $(q_1, q_2, i) \xrightarrow{a} (q_1', q_2', i')$ iff $q_1 \xrightarrow{a} q_1'$, $q_2 \xrightarrow{a} q_2'$, and one of the following conditions is satisfied:

(a) either $i = 0$, $i' = 1$, and $q_1' \in F_1$ or $i = i' = 0$ and $q_1' \notin F_1$,

(b) either $i = 1$, $i' = 2$, and $q_2' \in F_2$ or $i = i' = 1$ and $q_2' \notin F_2$, or

(c) $i = 2$ and $i' = 0$.

It remains to prove that $\mathcal{L}(\mathcal{B}) = \mathcal{L}_1 \cap \mathcal{L}_2$. We consider the inclusion $\mathcal{L}(\mathcal{B}) \subseteq \mathcal{L}_1 \cap \mathcal{L}_2$ and leave $\mathcal{L}(\mathcal{B}) \supseteq \mathcal{L}_1 \cap \mathcal{L}_2$ as exercise.

We introduce the following notation. For $i = 1, 2, 3$, let Π_i denote the i-th projection from $Q_1 \times Q_2 \times \{0,1,2\}$ on Q_1, resp. Q_2 and $\{0,1,2\}$.

First of all, it is not difficult to prove that ρ is a run of $\mathcal{B}$ iff $\Pi_i(\rho)$ is a run of $\mathcal{B}_i$, for $i = 1, 2$. It remains to prove that ρ is accepting iff $\Pi_1(\rho)$ and $\Pi_2(\rho)$ are.

To do so, we prove the following property of $\mathcal{B}$.

> For every run ρ of $\mathcal{B}$, and $i, j \in \omega$ with $i < j$, if $\Pi_3(\rho(i)) = 0$ and $\Pi_3(\rho(j)) = 2$ then there exists $k \in \omega$ with $\forall l \in \{i, \cdots, k-1\}$, $\Pi_3(\rho(l)) = 0$ and $\Pi_3(\rho(k)) = 1$.

By definition of $\mathcal{B}$, if $\Pi_3(\rho(j)) = 2$, then $\Pi_3(\rho(j-1)) = 1$. Let $k = \min\{l \mid i < l < j, \Pi_3(\rho(l)) = 1\}$. Then, $\forall l \in \{i, \cdots, k-1\}$, $\Pi_3(\rho(l)) = 0$ and $\Pi_3(\rho(k)) = 1$. □

Since by definition of $\mathcal{B}$, for every $i \in \omega$, if $\Pi_3(\rho(i)) = 0$ and $\Pi_3(\rho(i+1)) = 1$, then $\Pi_1(\rho(i+1)) \in F_1$, and if $\Pi_3(\rho(i)) = 2$, then $\Pi_2(\rho(i)) \in F_2$, we have $\mathit{inf}(\rho) \cap F \neq \emptyset$ iff $\mathit{inf}(\Pi_1(\rho)) \cap F_1 \neq \emptyset$ and $\mathit{inf}(\Pi_2(\rho)) \cap F_2 \neq \emptyset$.